

Selected problems from Wilde's book (Chapter 3)

3.2.1 Find (θ_i, ϕ_i) , $i \in \{+, -\}$, such that

$$|i\rangle = \cos\left(\frac{\theta_i}{2}\right)|0\rangle + e^{i\phi_i} \sin\left(\frac{\theta_i}{2}\right)|1\rangle.$$

Solution: We know that

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Write $|+\rangle = \cos\left(\frac{\theta_+}{2}\right)|0\rangle + e^{i\phi_+} \sin\left(\frac{\theta_+}{2}\right)|1\rangle$. Since $|+\rangle$ is normalized and $\langle +|0\rangle = \langle +|1\rangle = \frac{1}{\sqrt{2}}$, we have two equations:

$$\begin{cases} \cos\left(\frac{\theta_+}{2}\right) = \frac{1}{\sqrt{2}} \\ e^{-i\phi_+} \sin\left(\frac{\theta_+}{2}\right) = \frac{1}{\sqrt{2}} \end{cases}.$$

We can set $\theta_+ = \pi/2$ and $\phi_+ = 0$.

Similarly, for $|-\rangle$ we have

$$\begin{cases} \cos\left(\frac{\theta_-}{2}\right) = \frac{1}{\sqrt{2}} \\ e^{-i\phi_-} \sin\left(\frac{\theta_-}{2}\right) = -\frac{1}{\sqrt{2}} \end{cases}$$

We can set $\theta_- = \pi/2$ and $\phi_- = \pi$.

3.3.1 $T: \mathcal{H}_A \rightarrow \mathcal{H}_B$ is an isometry if and only if it is norm preserving.

Solution: Begin with sufficiency ($\Rightarrow$): let T be an isometry, i.e. $T^\dagger T = \mathbb{I}_A$. Choose an arbitrary vector $v \in \mathcal{H}_A$ and write $u = Tv$. We have

$$\|u\|_2 = \sqrt{\langle u, u \rangle_B} = \sqrt{\langle Tv, Tv \rangle} = \sqrt{\langle \underbrace{T^\dagger T}_{\mathbb{I}_A} v, v \rangle} = \sqrt{\langle v, v \rangle} = \|v\|_2.$$

Since v is arbitrary, this holds for all $v \in \mathcal{H}_A$. Thus T is norm-preserving.

We now prove necessity ($\Leftarrow$): let T be norm-preserving. This implies that for every $v \in H_A$:

$$\langle v, (T^*T - \mathbb{I}_A)v \rangle = 0.$$

All that's left is to show that if an operator $X: H_A \rightarrow H_A$ satisfies $\langle v, Xv \rangle = 0 \quad \forall v$, then $X = 0$. Set $A = T^*T - \mathbb{I}_A$.

For any u and v in H_A :

Polarization identity (antilinear 1st argument) The book uses the wrong one!

$$\begin{aligned} \langle x, Ay \rangle &= \frac{1}{4} (\langle x+Ay, x+Ay \rangle - \langle x-Ay, x-Ay \rangle - i\langle x+iAy, x+iAy \rangle + i\langle x-iAy, x-iAy \rangle) \\ &= \frac{1}{4} (\langle x, x \rangle + \langle x, Ay \rangle + \langle Ay, x \rangle + \langle Ay, Ay \rangle - \langle x, x \rangle + \langle x, Ay \rangle + \langle Ay, x \rangle - \langle Ay, Ay \rangle \\ &\quad - i\langle x, x \rangle - i\langle x, iAy \rangle - i\langle iAy, x \rangle - i\langle iAy, iAy \rangle + i\langle x, x \rangle - i\langle x, iAy \rangle - i\langle iAy, x \rangle + i\langle iAy, iAy \rangle) \\ &= \frac{1}{4} (\langle x, Ay \rangle + \langle y, Ay \rangle + \langle x, Ax \rangle + \langle y, Ax \rangle - \langle y, Ay \rangle - \langle x, Ax \rangle - \langle y, Ax \rangle + \langle x, Ay \rangle \\ &\quad - i\langle x, iAy \rangle - i\langle x, iAy \rangle) \\ &= \frac{1}{4} (\langle x+y, A(x+y) \rangle - \langle x-y, A(x-y) \rangle - 2\langle y, Ax \rangle - i\langle x, iAy \rangle - i\langle x, iAy \rangle) \\ &= \frac{1}{4} (\langle x+y, A(x+y) \rangle - \langle x-y, A(x-y) \rangle - i\langle x+iy, A(x+iy) \rangle + i\langle x-iy, A(x-iy) \rangle) \end{aligned}$$

Now, each term above is zero. Thus, for fixed y ,

$$\forall x \in H_A: \langle x, Ay \rangle = 0$$

Hence $Ay = 0$. Since y is arbitrary, we obtain $Ay = 0 \quad \forall y \in H_A$. Thus $A = 0$ and $T^*T = \mathbb{I}_A$!

3.3.10

Show that, under conjugation ($A \mapsto TAT^\dagger$) via the Hadamard gate, we have $HXH^\dagger = Z$ and $HZH^\dagger = X$.

Solution: $H = \frac{1}{\sqrt{2}}(|+\rangle\langle 0| + |-\rangle\langle 1|) = \frac{1}{\sqrt{2}}(|0\rangle\langle +| + |1\rangle\langle -|)$, $X = \frac{1}{\sqrt{2}}(|+\rangle\langle +| - |-\rangle\langle -|)$, $Z = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| - |1\rangle\langle 1|)$.

Hence:

$$\begin{aligned} \bullet HXH^\dagger &= HXH = \left(\frac{1}{\sqrt{2}}(|0\rangle\langle +| + |1\rangle\langle -|) \right) \left(\frac{1}{\sqrt{2}}(|+\rangle\langle +| - |-\rangle\langle -|) \right) \left(\frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |1\rangle\langle 1|) \right) \\ &= \left(\frac{1}{\sqrt{2}}(|0\rangle\langle +| - |1\rangle\langle -|) \right) \left(\frac{1}{\sqrt{2}}(|+\rangle\langle 0| + |-\rangle\langle 1|) \right) \\ &= \frac{1}{2}(|0\rangle\langle 0| - |1\rangle\langle 1|) = Z \end{aligned}$$

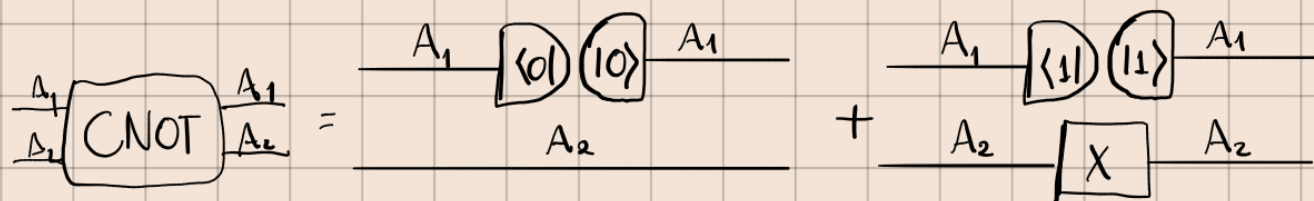
$$\begin{aligned} \bullet HZH^\dagger &= HZH = \left(\frac{1}{\sqrt{2}}(|+\rangle\langle 0| + |-\rangle\langle 1|) \right) \left(\frac{1}{\sqrt{2}}(|0\rangle\langle 0| - |1\rangle\langle 1|) \right) \left(\frac{1}{\sqrt{2}}(|0\rangle\langle +| + |1\rangle\langle -|) \right) \\ &= \left(\frac{1}{\sqrt{2}}(|+\rangle\langle 0| - |-\rangle\langle 1|) \right) \left(\frac{1}{\sqrt{2}}(|+\rangle\langle +| + |-\rangle\langle -|) \right) \\ &= \frac{1}{2}(|+\rangle\langle +| - |-\rangle\langle -|) = X \end{aligned}$$

In circuit notation:

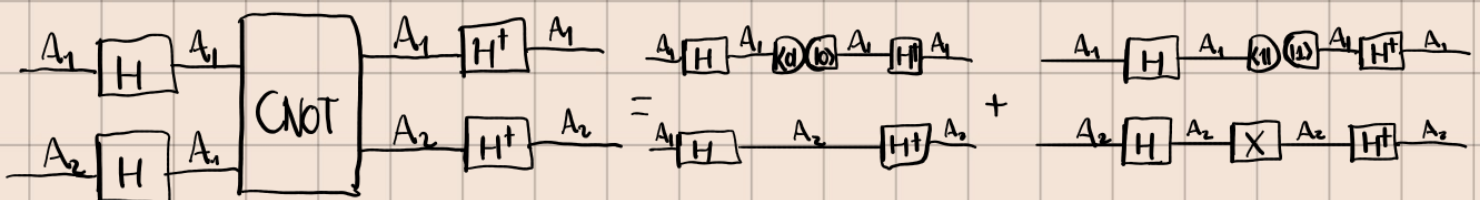


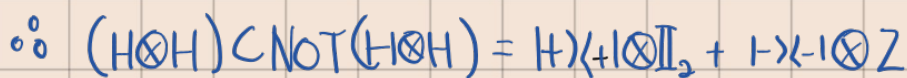
3.5.4

Conjugate a CNOT gate using $H \otimes H$. In diagrammatic representation:

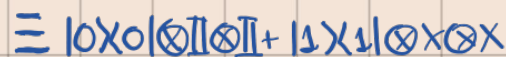


Thus:





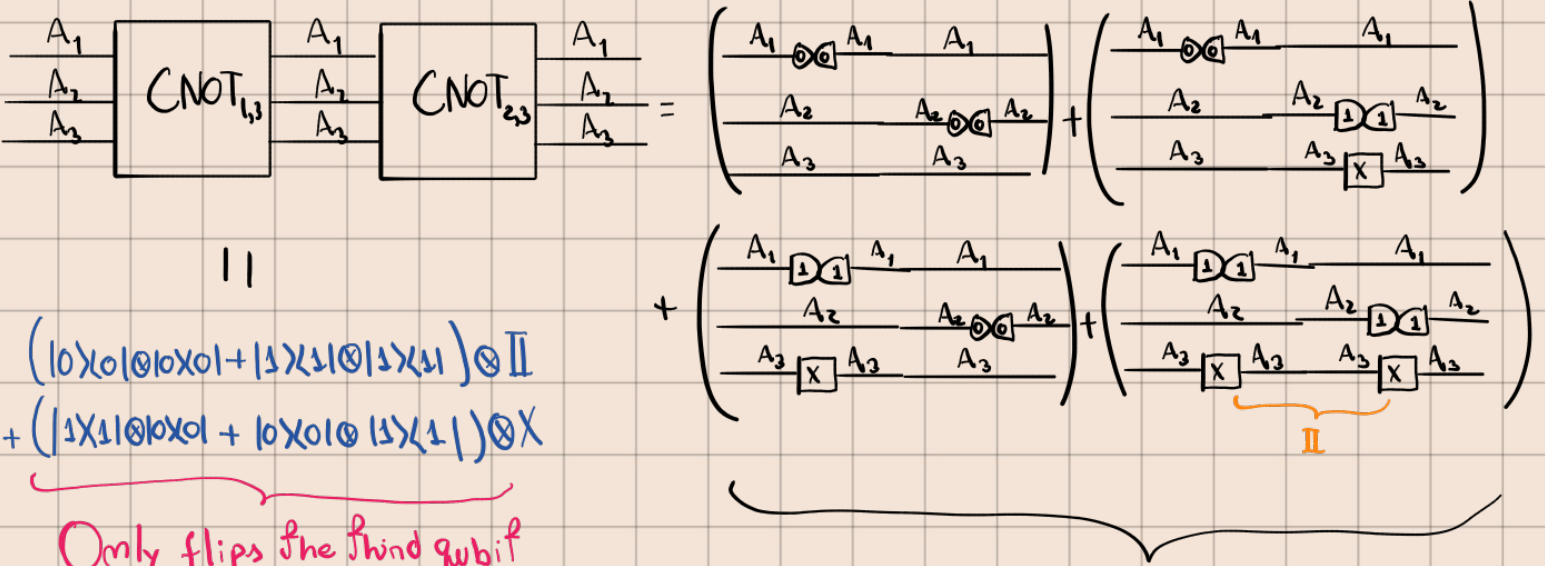
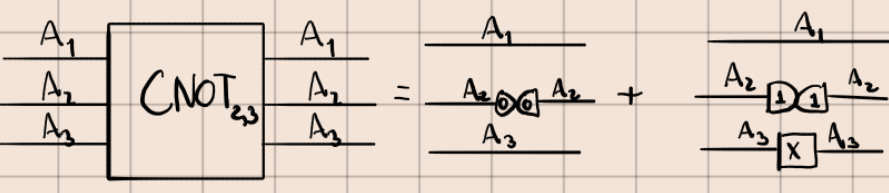
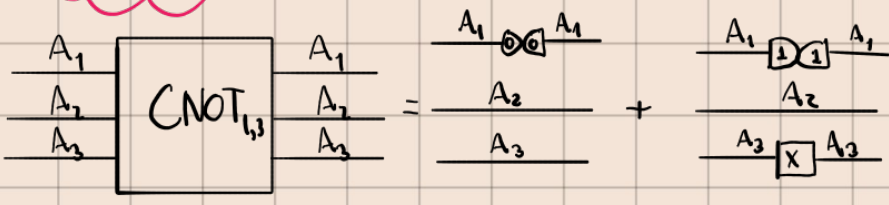
Show that $CNOT_{1,2}$ and $CNOT_{1,3}$ commute.



Clearly, if we computed $CNOT_{1,2} \circ CNOT_{1,3}$, the result would be the same

3.5.6

Show that $CNOT_{1,3}$ and $CNOT_{2,3}$ commute



$(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|) \otimes I$
 $+ (|1\rangle\langle 1| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1|) \otimes X$

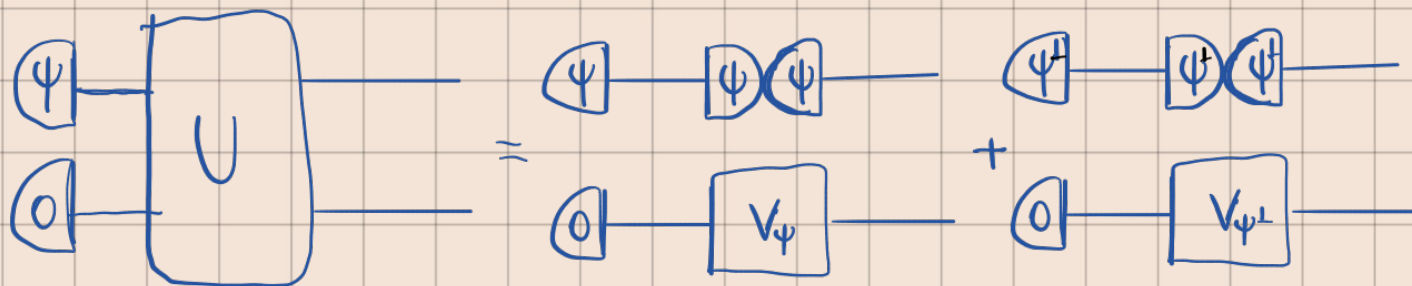
Only flips the third qubit if there's difference in parity

Clearly commutes: The diagram is preserved by dragging boxes to the right and to the left

Question: for n qubits and $n-1$ CNOT gates $CNOT_{1,n}, CNOT_{2,n}, \dots, CNOT_{n-1,n}$ does $(CNOT_{1,n}) \circ (CNOT_{2,n}) \circ \dots \circ (CNOT_{n-1,n})$ flips the n -th qubit depending on the parity of the first $n-1$?

3.5.7 Unitary measure-and-prepare.

Let $|\psi\rangle$ and $|\psi^\perp\rangle$ be orthogonal qubit states.



Let V_ψ and $V_{\psi^\perp}$ be the unitaries that take $|0\rangle$ to $|\psi\rangle$ and to $|\psi^\perp\rangle$ respectively. Set U as:

$$U = |\psi\rangle\langle\psi| \otimes V_\psi + |\psi^\perp\rangle\langle\psi^\perp| \otimes V_{\psi^\perp}.$$

Then

$$U|\psi\rangle \otimes |0\rangle = \langle\psi|\psi\rangle |\psi\rangle \otimes V_\psi|0\rangle + \langle\psi^\perp|\psi\rangle |\psi^\perp\rangle \otimes V_{\psi^\perp}|0\rangle \\ \stackrel{!}{=} |\psi\rangle \otimes |\psi\rangle$$

Similarly $U|\psi^\perp\rangle \otimes |0\rangle = |\psi^\perp\rangle \otimes |\psi^\perp\rangle.$

For concreteness: $V_\psi = \begin{pmatrix} \langle 0|\psi\rangle & \langle 0|\psi^\perp\rangle \\ \langle 1|\psi\rangle & \langle 1|\psi^\perp\rangle \end{pmatrix}, V_{\psi^\perp} = \begin{pmatrix} \langle 0|\psi^\perp\rangle & \langle 0|\psi\rangle \\ \langle 1|\psi^\perp\rangle & \langle 1|\psi\rangle \end{pmatrix}$

3.5.8 No-deletion theorem

Statement to be disproved: given $|I\rangle \in \mathcal{H}_B$, $\exists U$ s.t.

$$\forall |\psi\rangle \in \mathcal{H}_A: U|\psi\rangle \otimes |\psi\rangle \otimes |I\rangle = |\psi\rangle \otimes |0\rangle \otimes |F_\psi\rangle$$

Let $\{|\varphi_i\rangle\}_{i=1}^{d_A}$ be a basis in $\mathcal{H}_A$. The action of U on a state $|\varphi_i\rangle \otimes |\varphi_i\rangle \otimes |I\rangle$ is

$$U|\varphi_i\rangle \otimes |\varphi_i\rangle \otimes |I\rangle = |\varphi_i\rangle \otimes |R\rangle \otimes |F_{\varphi_i}\rangle,$$

Where $|R\rangle \in \mathcal{H}_A$ is a reference state and $|F_\psi\rangle$ is a state in $\mathcal{H}_B$ that depends on $|\psi\rangle$.

For $i \neq j$, set

$$U|\psi_i\rangle \otimes |\psi_j\rangle \otimes |I\rangle = |\Phi_{ij}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_A \otimes \mathcal{H}_B.$$

Now, the action of U on an arbitrary state $|\psi\rangle = \sum_{i=1}^{d_A} a_i |\psi_i\rangle$ is

$$\begin{aligned} U|\psi\rangle \otimes |\psi\rangle \otimes |I\rangle &= \sum_{i,j=1}^{d_A} a_i a_j U|\psi_i\rangle \otimes |\psi_j\rangle \otimes |I\rangle \\ &= \sum_{i=1}^{d_A} a_i^2 |\psi_i\rangle \otimes |R\rangle \otimes |F_{\psi_i}\rangle + \sum_{i < j} \frac{1}{\sqrt{2}} a_i a_j (|\Phi_{ij}\rangle + |\Phi_{ji}\rangle) \otimes |I\rangle \end{aligned}$$

But since U is a universal defecen, we also have

$$\begin{aligned} U|\psi\rangle \otimes |\psi\rangle \otimes |I\rangle &= |\psi\rangle \otimes |R\rangle \otimes |F_\psi\rangle \\ &= \sum_{i=1}^{d_A} a_i |\psi_i\rangle \otimes |R\rangle \otimes |F_\psi\rangle. \quad (\text{II}) \end{aligned}$$

For (I) to be equal to (II), the end state $|F_\psi\rangle$ must be a linear combination of each $|F_{\psi_i}\rangle$ and each $|\Phi_{ij}\rangle$ depends only on $|\psi_i\rangle$ and $|\psi_j\rangle$. More specifically:

$$\begin{aligned} |F_\psi\rangle &= \sum_{i=1}^{d_A} a_i |F_{\psi_i}\rangle \\ |\Phi_{ij}\rangle &= |\psi_i\rangle \otimes |R\rangle \otimes |F_{\psi_j}\rangle \end{aligned}$$

and

Given that $|\psi\rangle \otimes |R\rangle \otimes |F_\psi\rangle$ is normalized, we have

$$\sum_{i=1}^{d_A} |a_i|^2 \langle F_{\psi_i} | F_{\psi_i} \rangle = 1$$

But this is only possible if all $|F_{\psi_i}\rangle$ are normalized. Additionally, since $|F_\psi\rangle$ is normalized, we also have $\langle F_{\psi_i} | F_{\psi_j} \rangle = \delta_{ij}$.

Thus, $\{|\Phi_i\rangle\}_{i=1}^{d_A} \subset \mathcal{H}_B$ must be linearly independent. Hence, $V: \mathcal{H}_A \rightarrow \mathcal{H}_B$, $V|\Phi_i\rangle = |\Phi_i\rangle$ is an isometry and the action of U is

$$U|\Psi\rangle \otimes |\Psi\rangle \otimes |I\rangle = |\Psi\rangle \otimes |R\rangle \otimes V|\Psi\rangle. (*)$$

Without loss of generality, we can pick $|R\rangle = V^\dagger |I\rangle$. Then by applying $V \otimes V^\dagger$ on the last two systems:

$$(\mathbb{I}_A \otimes V \otimes V^\dagger) U |\Psi\rangle \otimes |\Psi\rangle \otimes |I\rangle = |\Psi\rangle \otimes |I\rangle \otimes |\Psi\rangle. (**)$$

Eq. (*) represents the fact that U only encodes $|\Psi\rangle$ in system B , and eq. (**) shows how to retrieve $|\Psi\rangle$. Hence, U does not delete information, it just reshuffles it in a recoverable manner. U is implemented by swapping the systems and applying $V^\dagger \otimes V$.

Conclusion: if swapping is allowed, U exists but does not delete information. If swapping is not allowed, U does not exist. So, information is never deleted!

3.6.5

Basic calculations involving Bell states

First, recall that

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$

For a pair of bits $(z, x) \in \{0, 1\}^{x_2}$, we set

$$|\Phi^{zx}\rangle = (Z^z X^x \otimes \mathbb{I}_2) |\Phi^+\rangle$$

1) The 4 states $|\phi^{00}\rangle, |\phi^{01}\rangle, |\phi^{10}\rangle, |\phi^{11}\rangle$ form an orthonormal basis:

First, for any $x \in \{0, 1\}$: $X_A^x |\phi^+\rangle = \frac{1}{\sqrt{2}} (|+\rangle|+\rangle + (-1)^x |-\rangle|-\rangle)$.

Thus:

$$\begin{aligned} \langle \phi^{\beta x} | \phi^{\beta' x'} \rangle &= \langle \phi^+ | X_A^x Z_A^{\beta} Z_A^{\beta'} X_A^{x'} | \phi^+ \rangle \\ &= \langle \phi^+ | X_A^x Z_A^{\beta \oplus \beta'} X_A^{x'} | \phi^+ \rangle \\ &= \frac{1}{2} [(\langle ++ | + (-1)^x \langle -- |) Z_A^{\beta \oplus \beta'} (|++\rangle + (-1)^{x'} |--\rangle)] \\ &= \frac{1}{2} (\langle ++ | Z_A^{\beta \oplus \beta'} | ++ \rangle + (-1)^{x'} \langle ++ | Z_A^{\beta \oplus \beta'} | -- \rangle + (-1)^x \langle -- | Z_A^{\beta \oplus \beta'} | ++ \rangle + (-1)^{x \oplus x'} \langle -- | Z_A^{\beta \oplus \beta'} | -- \rangle) \\ &= \frac{1}{2} (\langle + | Z_A^{\beta \oplus \beta'} | + \rangle + (-1)^{x \oplus x'} \langle - | Z_A^{\beta \oplus \beta'} | - \rangle) \end{aligned}$$

Now, $Z_A |+\rangle_A = |-\rangle_A$ and $Z_A |-\rangle_A = |+\rangle_A$. Thus if $\beta \oplus \beta' = 1$, i.e. $\beta \neq \beta'$, all terms will be zero whereas if $\beta \oplus \beta' = 0$, i.e. $\beta = \beta'$, we are left with

$$\frac{1}{2} (1 + (-1)^{x \oplus x'}) = \delta_{xx'}$$

Thus

$$\langle \phi^{\beta x} | \phi^{\beta' x'} \rangle = \delta_{\beta\beta'} \delta_{xx'}$$

3.7.1 - 3.7.5 Shift and phase operators on qudits

Let $\mathcal{H}$ be d -dimensional. If $\{|j\rangle\}_{j=0}^{d-1}$ is the canonical basis, we construct the following operator:

$$X(x) |j\rangle = |x \oplus j\rangle \quad (\text{Shift operator})$$

where $x \in \mathbb{Z}$ and $x \oplus j \equiv (x+j) \bmod(d)$.

We have

$$\begin{aligned} X(-x)X(x)|y\rangle &= X(-x)|x\oplus y\rangle \\ &\stackrel{!}{=} |-x\oplus(x\oplus y)\rangle \\ &\stackrel{!}{=} |(-x\oplus x)\oplus y\rangle \\ &\stackrel{!}{=} |0\oplus y\rangle \\ &\stackrel{!}{=} |y\rangle. \end{aligned}$$

Similarly, $X(x)X(-x)|y\rangle = |y\rangle$. Thus $X(-x)$ is both a right and left inverse of $X(x)$.

The matrix elements of $X(x)$ are

$$\langle i|X(x)|j\rangle = \langle i|x\oplus j\rangle = \delta_{i, x\oplus j}$$

For example, take $d=3$ and $x=1$ on $x=2$:

$$X(1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad X(2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Another operation we can define is

$$Z(z)|j\rangle = e^{i2\pi z j/d} |j\rangle \quad (\text{Phase operation})$$

Take $d=2$ and $z=1$ ($z=0$ is the identity):

$$Z(1)|0\rangle = e^{i2\pi 1 \cdot 0/2} |0\rangle = |0\rangle$$

$$Z(1)|1\rangle = e^{i2\pi 1 \cdot 1/2} |1\rangle = -|1\rangle$$

Thus $Z(1) = \sigma_z \equiv Z$.

For $x, z \in \{0, d-1\}$, the d^2 operations $\{X(x)Z(z)\}$ are called Heisenberg-Weyl operations.

The inverse of $Z(\gamma)$ is $Z(-\gamma)$, that is obvious. The matrix elements of $Z(\gamma)$ are

$$\langle i | Z(\gamma) | j \rangle = e^{i2\pi\gamma j/d} \langle i | j \rangle = e^{i2\pi\gamma j/d} \delta_{ij}.$$

In particular, $Z(\gamma)$ has eigenvalues $e^{i2\pi\gamma j/d}$.
For instance, if $d=3$ and $\gamma=1$ or 2 :

$$Z(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i2\pi/3} & 0 \\ 0 & 0 & e^{i4\pi/3} \end{pmatrix}; Z(2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i4\pi/3} & 0 \\ 0 & 0 & e^{i8\pi/3} \end{pmatrix}$$

3.7.6 Eigenstates of $X(1)$.

Suppose $|\psi\rangle$ is an eigenstate of $X(1)$. Write $|\psi\rangle = \sum_{j=0}^{d-1} a_j |j\rangle$. Then

$$\begin{aligned} X(1)|\psi\rangle &= \lambda |\psi\rangle \Rightarrow \sum_{j=0}^{d-1} a_j |1 \oplus j\rangle = \lambda \sum_{j=0}^{d-1} a_j |j\rangle \\ &\quad \parallel \\ \sum_{j=0}^{d-1} a_{-1 \oplus j} |j\rangle &\Rightarrow \boxed{a_{-1 \oplus j} = \lambda a_j} \quad (*) \end{aligned}$$

$$j=0: a_{d-1} = \lambda a_0$$

$$j=1: a_0 = \lambda a_1$$

$$j=2: a_1 = \lambda a_2$$

$$\vdots$$

$$j=d-1: a_{d-2} = \lambda a_{d-1}$$

$$a_{d-1} = \lambda^{d-1} a_0$$

$$\leadsto a_1 = \lambda^{-1} a_0 \leadsto \lambda \left[\left(\frac{1}{\lambda} \right)^d - 1 \right] = 0$$

$$a_{d-2} = \lambda^{2-d} a_0$$

Thus we have $\lambda=0$, $\lambda=1$ or λ^{-1} is a d -th root of unity.
If $\lambda=0$, then $|\psi\rangle=0$ (due to (*)), but that can't happen because eigenvectors are non-zero.

If $\lambda = 1$, then $a_1 = a_2 = \dots = a_{d-1} = a_0$. Thus

$$|\psi_{\lambda=1}\rangle = a_0 \sum_{j=0}^{d-1} |j\rangle \Rightarrow \boxed{a_0 = \frac{1}{\sqrt{d}}} \text{ and } \boxed{|\tilde{0}\rangle = |\psi_{\lambda=1}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle}$$

Now, the remaining possible λ 's are the other roots of unity:

$$\boxed{\lambda_\ell = e^{-i2\pi\ell/d}, \ell \in \{1, \dots, d-1\}}$$

So for each ℓ we rewrite $|\psi_{\lambda_\ell}\rangle$ as $|\tilde{\ell}\rangle$. We have that

$$\boxed{|\tilde{\ell}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i2\pi\ell j/d} |j\rangle}$$

For arbitrary $x \in \{0, \dots, d-1\}$, we have

$$\begin{array}{l} j=0: a_0 = \lambda a_x \\ j=1: a_1 = \lambda a_{x \oplus 1} \\ j=2: a_2 = \lambda a_{x \oplus 2} \\ \vdots \\ j=d-x-1: a_{d-x-1} = \lambda a_{d-1} \\ j=d-x: a_{d-x} = \lambda a_0 \\ j=d-x+1: a_{d-x+1} = \lambda a_1 \\ j=d-x+2: a_{d-x+2} = \lambda a_2 \\ \vdots \\ j=d-1: a_{d-1} = \lambda a_{x-1} \end{array} \quad \begin{array}{l} \text{0, 1, 2, 3, 4, 5, 6, 7, 8, 9} \\ \text{d=10} \\ \text{x=1, steps: 3} \\ \text{x=2, steps: 2} \\ \text{x=3, steps: 4} \\ \text{loop: 10} \\ \text{loop: 5} \\ \text{loop: 5} \end{array}$$

Para loopar: necessário $\frac{m}{x}$ passos.
onde m é o número de elementos
 $x \in d$

$$a_0 = \lambda^{m/x} a_0 \Rightarrow \lambda^{m/x} = 1?$$

~o Tny handed

For arbitrary shift $x \in \{0, \dots, d-1\}$, just notice that

$$X(x) = \underbrace{X(1) \dots X(1)}_{x \text{ times}}$$

Thus, if $|\tilde{l}\rangle$ is an eigenstate of $X(1)$ with eigenvalue λ_l , then $|\tilde{l}\rangle$ is also an eigenstate of $X(x)$, but with eigenvalue $(\lambda_l)^x$.

3.7.7 Dimension 2 shift operator $X(1)$.

The eigenvalues of $X(1)$ in $d=2$ are $\lambda_0 = e^{-i2\pi \cdot 0/2} = 1$ and $\lambda_1 = e^{-i2\pi/2} = -1$.

The eigenstates are

$$|\tilde{0}\rangle = \frac{1}{\sqrt{2}} \left(e^{i2\pi \cdot 0/2} |0\rangle + e^{i2\pi \cdot 1/2} |1\rangle \right) = |+\rangle$$

$$|\tilde{1}\rangle = \frac{1}{\sqrt{2}} \left(e^{i2\pi \cdot 0/2} |0\rangle + e^{i2\pi \cdot 1/2} |1\rangle \right) = |-\rangle$$

3.7.8 The qudit Fourier transform is the map given

by $F = \sum_{j=0}^{d-1} |\tilde{j}\rangle\langle j|$, i.e. $|j\rangle \xrightarrow{F} \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{i2\pi jk/d} |k\rangle$

• Conjugating $X(x)$:

$$\begin{aligned} F X(x) F^\dagger &= \sum_{j,j'} |\tilde{j}\rangle\langle j| X(x) |j'\rangle\langle \tilde{j}'| \\ &= \sum_{j,j'} \delta_{j,j \oplus x} |\tilde{j}\rangle\langle \tilde{j}'| \\ &= \sum_{j'=0}^{d-1} |\tilde{j \oplus x}\rangle\langle \tilde{j}'| \\ &= \sum_{j'=0}^{d-1} \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} \frac{1}{d} e^{i2\pi(j \oplus x)k/d} e^{-i2\pi j'l/d} |k\rangle\langle l| \\ &= \sum_{j',k,l=0}^{d-1} \frac{1}{d} e^{i2\pi[(j \oplus x)k - j'l]/d} |k\rangle\langle l| = \sum_{k=0}^{d-1} e^{i2\pi xk/d} |k\rangle\langle k| = Z(x) \end{aligned}$$

• Conjugating $Z(z)$:

$$\begin{aligned} FZ(z)F^\dagger &= \sum_{j,j'=0}^{d-1} |\tilde{j}\rangle\langle j| Z(z) |j'\rangle\langle \tilde{j}'| \\ &= \sum_{j,j'=0}^{d-1} \delta_{jj'} e^{i2\pi jj'/d} |\tilde{j}\rangle\langle \tilde{j}'| \\ &= \sum_{j=0}^{d-1} e^{-i2\pi(-j)j/d} |\tilde{j}\rangle\langle \tilde{j}| = \boxed{X(-z)} \end{aligned}$$

{3.7.9} Commutation of $X(z)$ and $Z(z)$.

$$Z(z)X(x)|j\rangle = Z(z)|j\oplus x\rangle = e^{i2\pi j(j\oplus x)/d} |j\oplus x\rangle$$

$$\begin{aligned} X(x)Z(z)|j\rangle &= X(x) e^{i2\pi jj/d} |j\rangle \\ &= e^{i2\pi jj/d} |j\oplus x\rangle \end{aligned}$$

$$\Rightarrow [Z(z), X(x)]|j\rangle = (e^{i2\pi jx} - 1) e^{i2\pi jj/d} |j\oplus x\rangle$$

$$\text{on } X(x)Z(z) = e^{-i2\pi xj/d} Z(z)X(x)$$

$$\begin{aligned} X(x_1)Z(z_1)X(x_2)Z(z_2) &= X(x_1) e^{i2\pi x_2 z_1/d} X(x_2)Z(z_1)Z(z_2) \\ &= e^{i2\pi x_2 z_1/d} X(x_2)X(x_1)Z(z_2)Z(z_1) \\ &= e^{i2\pi(x_2 z_1 - x_1 z_2)} X(x_2)Z(z_2)X(x_1)Z(z_1) \end{aligned}$$

3.7.10

Measuring $Z(1)$ and $X(1)$. For any $x \neq 0 \pmod{d}$ and any $z \neq 0 \pmod{d}$, $Z(z)$ and $X(x)$ are not Hermitian: Z has complex eigenvalues, and so does $X(x)$.

We measure them indirectly. For instance, we can measure $Z(1)$

Using

$$M_{Z(1)} = \sum_{j=0}^{d-1} j |j\rangle\langle j|$$

As for $X(1)$:

$$M_{X(1)} = \sum_{j=0}^{d-1} j |\tilde{j}\rangle\langle \tilde{j}|$$

Let $|\psi\rangle = \sum_{j=0}^{d-1} a_j |j\rangle$. In the $|\tilde{j}\rangle$ basis:

$$\begin{aligned} |\psi\rangle &= \sum_{j=0}^{d-1} \tilde{a}_j |\tilde{j}\rangle = \sum_{j=0}^{d-1} \langle \tilde{j} | \psi \rangle |\tilde{j}\rangle = \sum_{j=0}^{d-1} \langle \tilde{j} | F^\dagger F | \psi \rangle |\tilde{j}\rangle \\ &= \sum_{j=0}^{d-1} \frac{1}{\sqrt{d}} \left(\sum_{k=0}^{d-1} e^{-i2\pi jk/d} \langle \tilde{k} | F | \psi \rangle \right) |\tilde{j}\rangle = \sum_{j=0}^{d-1} \frac{1}{\sqrt{d}} \left(\sum_{k=0}^{d-1} e^{-i2\pi jk/d} a_k \right) |\tilde{j}\rangle \end{aligned}$$

Now, the expectation value of $M_{X(1)}$ is:

$$\langle M_{X(1)} \rangle = \langle \psi | M_{X(1)} | \psi \rangle = \frac{1}{d} \sum_{j_1=0}^{d-1} \sum_{j_2=0}^{d-1} \sum_{k_1=0}^{d-1} \sum_{k_2=0}^{d-1} a_{k_1}^* a_{k_2} e^{i2\pi j_1 k_1/d} e^{-i2\pi j_2 k_2/d} \langle \tilde{j}_1 | M_{X(1)} | \tilde{j}_2 \rangle$$

$$= \frac{1}{d} \sum_{j_1=0}^{d-1} \sum_{j_2=0}^{d-1} \sum_{k_1=0}^{d-1} \sum_{k_2=0}^{d-1} a_{k_1}^* a_{k_2} e^{i2\pi j_1 k_1/d} e^{-i2\pi j_2 k_2/d} j_2 \langle \tilde{j}_1 | \tilde{j}_2 \rangle$$

$$= \frac{1}{d} \sum_{j_1=0}^{d-1} \sum_{k_1=0}^{d-1} \sum_{k_2=0}^{d-1} a_{k_1}^* a_{k_2} e^{i2\pi j_1 k_1/d} e^{-i2\pi j_2 k_2/d} j_2$$

$$\begin{aligned}
&= \frac{1}{d} \sum_{\delta=0}^{d-1} \sum_{k_1=0}^{d-1} \sum_{k_2=0}^{d-1} a_{k_1}^* a_{k_2} \delta \exp[i 2\pi (k_1 - k_2) \delta / d] \\
&= \frac{1}{d} \sum_{\delta=0}^{d-1} \delta \left(\sum_{k_1=0}^{d-1} a_{k_1}^* e^{i 2\pi \delta k_1 / d} \right) \left(\sum_{k_2=0}^{d-1} a_{k_2} e^{-i 2\pi \delta k_2 / d} \right) \\
&= \frac{1}{d} \sum_{\delta=0}^{d-1} \delta \left| \sum_{k=0}^{d-1} a_k e^{-i 2\pi \delta k / d} \right|^2
\end{aligned}$$

3.7.11 Qudit analogue of Bell states

$\forall x \in \{0, \dots, d-1\}, \forall z \in \{0, \dots, d-1\}$:

$$|\Phi^{xz}\rangle_{AB} = (X(x)Z(z))_A \otimes \mathbb{I}_B |\Phi\rangle_{AB}$$

For arbitrary x, x', z, z' :

$$\begin{aligned}
\langle \Phi^{x'z'} | \Phi^{xz} \rangle_{AB} &= \langle \Phi | (Z(z')^\dagger X(x')^\dagger)_A \otimes \mathbb{I}_B \cdot (X(x)Z(z))_A \otimes \mathbb{I}_B | \Phi \rangle_{AB} \\
&= \langle \Phi | [Z(z') X(-x) X(x') Z(z')]_A \otimes \mathbb{I}_B | \Phi \rangle_{AB} \\
&= \langle \Phi | [Z(z') X(\widetilde{x \ominus x'}) Z(z')]_A \otimes \mathbb{I}_B | \Phi \rangle_{AB} \quad (x \ominus x' = (x' + (-x))_{\text{mod } d}) \\
&= \langle \Phi | [Z(z' \oplus z) X(x' \ominus x)]_A \otimes \mathbb{I}_B | \Phi \rangle e^{-i 2\pi (x \ominus x') z' / d}
\end{aligned}$$

$$X(x \ominus x') |z\rangle_A = |z \oplus (x \ominus x')\rangle_A$$

$$Z(z' \oplus z) X(x \ominus x') |z\rangle_A = e^{i 2\pi (z' \oplus z) \cdot z \ominus x \ominus x' / d} |z \oplus (x \ominus x')\rangle_A$$

Thus

$$\begin{aligned}
 \langle \Phi | (Z(z\theta z) X(x'\theta x)) \otimes \Pi_B | \Phi \rangle &= \frac{1}{d} \sum_{\delta, k=0}^{d-1} \left[e^{-i2\pi(x'\theta x) Z(z\theta z)/d} e^{i2\pi(z\theta z) \cdot \delta \theta x'\theta x/d} \right. \\
 &\quad \left. \langle k | \delta \theta x'\theta x \rangle_A \langle k | \delta \rangle_B \right] \\
 &= \frac{1}{d} \sum_{\delta, k=0}^{d-1} e^{-i2\pi(x'\theta x) Z(z\theta z)/d} e^{i2\pi(z\theta z) \cdot \delta \theta x'\theta x/d} \overbrace{\delta_{\delta, \delta \theta x'\theta x}}^{\delta_{\delta, \delta \theta x'\theta x}} \delta_{k, \delta \theta x'\theta x} \delta_{k, \gamma} \\
 &= \frac{\exp(-i2\pi(x'\theta x) Z(z\theta z)/d)}{d} \sum_{\delta=0}^{d-1} \delta_{\delta \theta x'\theta x} \exp(i2\pi(z\theta z) \cdot \delta \theta x'\theta x/d) \\
 \delta = \delta \theta (x'\theta x) &\Leftrightarrow x'\theta (x) \equiv 0 \pmod{d} \\
 &\Downarrow \\
 (x'\theta x)(\delta \theta z) &= 0 \pmod{d} \\
 &\Downarrow \\
 &= \frac{1}{d} \sum_{\delta=0}^{d-1} \exp(i2\pi(z\theta z) \delta/d) \delta_{x'\theta x, 0} \\
 &= \delta_{0, z\theta z} \delta_{x'\theta x, 0}
 \end{aligned}$$

Since x' and x are in $\{0, \dots, d-1\}$, we have $-(d-1) \leq x' - x \leq d-1$, thus $x'\theta x = (x' - x) \pmod{m} = 0 \Rightarrow x' - x = 0 \Rightarrow x' = x$

Similarly, $\delta'\theta \delta = 0 \Rightarrow \delta' = \delta$

Thus:

$$\langle \Phi^{x\delta} | \Phi^{x'\delta'} \rangle_{AB} = \delta_{x, x'} \delta_{\delta, \delta'}. \quad (\text{Orthogonality})$$

$$\begin{aligned}
 \sum_{x, \delta} |\Phi^{x\delta}\rangle \langle \Phi^{x\delta}|_{AB} &= \sum_{x, \delta=0}^{d-1} (X(\omega Z(z)) \otimes \Pi_B | \Phi \rangle \langle \Phi | (Z(z) X(x)) \otimes \Pi_B) \\
 &= \frac{1}{d} \sum_{x, \delta=0}^{d-1} \sum_{k=0}^{d-1} \left([X(x) Z(z) | \delta \rangle] \otimes | \delta \rangle \right) \left(\langle k | Z(z) X(x) \rangle \otimes \langle k | \right)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{d} \sum_{x,j=0}^{d-1} \sum_{k=0}^{d-1} \left(\chi(x) \chi(j) |j\rangle \chi(k) \chi(-j) \chi(-x) \right) \otimes |j\rangle \langle k| \\
&= \frac{1}{d} \sum_{x,j=0}^{d-1} \sum_{k=0}^{d-1} e^{i2\pi jx/d} e^{-i2\pi jk/d} (|j \oplus x\rangle \langle k \oplus x|) \otimes |j\rangle \langle k| \\
&= \sum_{x=0}^{d-1} \sum_{j,k=0}^{d-1} \underbrace{\left(\frac{1}{d} \sum_{j=0}^{d-1} e^{i2\pi(j-k)x/d} \right)}_{\delta_{jk}} |j \oplus x\rangle \langle k \oplus x| \otimes |j\rangle \langle k| \\
&= \sum_{x=0}^{d-1} \sum_{j=0}^{d-1} |j \oplus x\rangle \langle j \oplus x| \otimes |j\rangle \langle j| = \sum_{j=0}^{d-1} \underbrace{\left(\sum_{x=0}^{d-1} \chi(x) |j\rangle \chi(j) \chi^\dagger(x) \right)}_{\Pi_d} \otimes |j\rangle \langle j| \\
&= \Pi_d \otimes \sum_{j=0}^{d-1} |j\rangle \langle j| = \Pi_d \otimes \Pi_d
\end{aligned}$$

3.7.12 Transpose Trick

If M is any operation in $\mathcal{H}$ and $|\Phi\rangle$ is the maximally entangled state $\frac{1}{\sqrt{d}} \sum |ii\rangle \in \mathcal{H} \otimes \mathcal{H}$, then

$$(M \otimes \Pi_d) |\Phi\rangle = (\Pi_d \otimes M^T) |\Phi\rangle, \quad \text{where } \langle i|M^T|j\rangle = M_{ji}$$

Proof:

$$\begin{aligned}
M \otimes \Pi_d |\Phi\rangle &= \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} M|i\rangle \otimes |i\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \left(\sum_{j=0}^{d-1} \langle j|M|i\rangle \right) |j\rangle \otimes |i\rangle \\
&= \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle \otimes \left(\sum_{i=0}^{d-1} \overbrace{\langle j|M|i\rangle}^{i \cdot M^T \cdot j} |i\rangle \right) \\
&= \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle \otimes \left(\sum_{i=0}^{d-1} |i\rangle \langle i|M^T|j\rangle \right) \\
&= \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle \otimes M^T|j\rangle = \Pi_d \otimes M^T |\Phi\rangle
\end{aligned}$$

3.8.1 The Schmidt decomposition Theorem

We just have to prove the following and the theorem follows:

Theorem (Singular values decomposition). Let $T: V \rightarrow W$ be a linear operator between finite-dimensional inner product vector spaces over $\mathbb{C}$. There exist an orthonormal basis $(v_1, \dots, v_m)$ of V , an orthonormal basis $(w_1, \dots, w_m)$ of W and non-negative numbers $\sigma_1, \dots, \sigma_m$ such that

$$Tv_i = \sigma_i w_i \quad (1 \leq i \leq \text{rank}(T)) \quad \text{and} \quad T(v_i) = 0 \quad (i > \text{rank}(T)).$$

In particular, for any fixed orthonormal bases $(e_i)_{i=1}^n$ of V and $(f_j)_{j=1}^m$ of W , there are unitaries $P e_i = v_i$ and $Q f_j = w_j$ and an operator $\Sigma: V \rightarrow W$ satisfying

$$\Sigma e_i = \begin{cases} \sigma_i f_i & , i = 1, \dots, \text{rank}(T) \\ 0 & , i > \text{rank}(T) \end{cases}$$

such that

$$T = Q \Sigma P^\dagger$$

Proof. Given $T: V \rightarrow W$, the operator $T^\dagger T$ is positive and it has a unique positive square root S , i.e., S is Hermitian with non-negative eigenvalues $\sigma_1, \dots, \sigma_m$ (not all different) and $S^2 = T^\dagger T$.

W.l.o.g. let $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_m$, and set $\pi = \text{rank}(T)$. There are precisely π non-zero σ_i 's. To see that, notice that $\text{Ker}(S)$ and $\text{Ker}(T)$ are the same. Since the following holds

$$\dim \text{Ker}(S) + \dim \text{Ran}(S) = \dim V, \quad (*)$$

$$\dim \text{Ker}(T) + \dim \text{Ran}(T) = \dim V,$$

We have $\dim \text{Ran}(S) = \dim \text{Ran}(T) = \text{rank}(T) = \pi$.

Let $\{v_1, \dots, v_m\}$ be a basis of V s.f.

$$Sv_i = \sigma_i v_i \quad \forall i \in \{1, \dots, m\}.$$

The kernel of S is precisely the zero eigenspace of S , thus it is spanned by k vectors from $\{v_1, \dots, v_m\}$. Consequently, due to (*), $\text{Ran}(S)$ is spanned by the $m-k$ vectors for which $Sv_i = \sigma_i v_i$ with $\sigma_i > 0$. By the previous observation, $m-k=r$.

Now, for $i=1, \dots, r$, set

$$w_i = \frac{1}{\sigma_i} T v_i.$$

We have

$$\langle w_i, w_j \rangle = \frac{1}{\sigma_i \sigma_j} \langle T^T T v_i, v_j \rangle = \frac{\sigma_i}{\sigma_j} \delta_{ij} = \delta_{ij}$$

Thus $\{w_1, \dots, w_r\} \subset W$ is an orthonormal set. Additionally,

$$T v_i = \sigma_i w_i.$$

For $i > r$, $v_i \in \ker(T)$, thus

$$T v_i = 0,$$

as desired. If necessary, we can extend $\{w_1, \dots, w_r\}$ to an orthonormal basis $\{w_1, \dots, w_r, w_{r+1}, \dots, w_m\}$ in W , with $w_{r+1}, \dots, w_m \in \text{Ran}(T)^\perp$.

Now, fix arbitrary bases $\{e_i\}_{i=1}^m \subset V$ and $\{f_j\}_{j=1}^m \subset W$. Construct the following operators:

$$P e_i = v_i \quad \text{and} \quad Q f_j = w_j.$$

They satisfy

$$\langle (P^T P - I_A) v, v \rangle_v = 0 \quad \forall v \in V$$

$$\langle (Q^T Q - I_B) w, w \rangle_w = 0 \quad \forall w \in W$$

Thus $P^*P = I_A$ and $Q^*Q = I_B$. From this, we conclude that P and Q are unitary.

Now, define $\Sigma: V \rightarrow W$ simply as

$$\Sigma(e_i) = \sigma_i f_i \quad \forall i \in \{1, \dots, n\}$$

$$\Sigma(e_i) = 0 \quad \forall i > n.$$

Then for arbitrary $v \in V$:

$$\begin{aligned} (Q\Sigma P^*)v &= Q\Sigma P^*\left(\sum_{i=1}^m \langle v, v_i \rangle v_i\right) \\ &= Q\Sigma\left[\sum_{i=1}^m \langle v, v_i \rangle P^*v_i\right] \\ &= Q\Sigma\left[\sum_{i=1}^m \langle v, v_i \rangle e_i\right] \\ &= Q\left(\sum_{i=1}^m \langle v, v_i \rangle \Sigma(e_i)\right) \\ &= Q\left(\sum_{i=1}^n \langle v, v_i \rangle \sigma_i f_i\right) \\ &= \sum_{i=1}^n \sigma_i \langle v, v_i \rangle Q(f_i) \\ &= \sum_{i=1}^n \sigma_i \langle v, v_i \rangle w_i \\ &= Tv. \end{aligned}$$

Hence: $\boxed{T = Q\Sigma P^*}$

$G: \mathcal{H}_B \rightarrow \mathcal{H}_A$, fix orthonormal bases $\{|\psi_j\rangle_A\}_{j=1}^{m_A}$ and $\{|\varphi_i\rangle_B\}_{i=1}^{m_B}$. By the theorem, there are orthonormal bases $\{|\Phi_i\rangle_B\}_{i=1}^{m_B}$, $\{|\Psi_j\rangle_A\}_{j=1}^{m_A}$ and $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$

$$\begin{cases} V^\dagger |\varphi_i\rangle_B = |\Phi_i\rangle_B; \\ U |\psi_j\rangle_A = |\Psi_j\rangle_A; \\ \Sigma |\varphi_i\rangle_B = \sigma_i |\Psi_i\rangle_A \quad i = 1, \dots, r = \text{rank}(G); \\ G = U \Sigma V. \quad (**) \end{cases}$$

In matrix rep. $[G]_{jk} = \langle \psi_j, G \varphi_k \rangle_A = \alpha_{jk}$, $[U]_{ji} = \langle \psi_j, U \varphi_i \rangle_A = u_{ji}$
 $[V]_{ik} = [V^\dagger]_{ki}^* = \langle \varphi_k, V^\dagger \varphi_i \rangle^* = \langle V^\dagger \varphi_i, \varphi_k \rangle = \langle \varphi_i, V \varphi_k \rangle$.
 $[\Sigma]_{il} = \langle \varphi_i, \Sigma \varphi_l \rangle_A = \langle \varphi_i, \sigma_l \varphi_l \rangle = \sigma_l \delta_{il}$ for $l \leq r$, 0 otherwise

In the $\{|\varphi_k\rangle_B\}, \{|\psi_j\rangle_A\}$ basis: $|\mu\rangle_B = \sum_{k=1}^{m_B} c_k |\varphi_k\rangle_B$

$$G|\mu\rangle_B = \sum_{k=1}^{m_B} c_k G|\varphi_k\rangle_B = \sum_{k=1}^{m_B} \sum_{j=1}^{m_A} c_k G_{jk} |\psi_j\rangle_A$$

$$(U\Sigma V)|\mu\rangle_B = \sum_{k=1}^{m_B} c_k U\Sigma V|\varphi_k\rangle_B = \sum_{k=1}^{m_B} \sum_{j=1}^{m_A} c_k (U\Sigma V)_{jk} |\psi_j\rangle_A$$

$$\Rightarrow G_{jk} = (U\Sigma V)_{jk} \Rightarrow \underline{\alpha_{jk}} = \sum_{i=1}^{m_A} \sum_{l=1}^{m_B} u_{ji} \Sigma_{il} v_{lk} = \boxed{\sum_{i=1}^r u_{ji} \sigma_i v_{ik}}$$

Since α_{jk} uniquely defines a state $|\xi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ via

$$|\xi\rangle_{AB} = \sum_{j,k=1}^{m_A, m_B} \alpha_{jk} |\psi_j\rangle_A \otimes |\varphi_k\rangle_B$$

We obtain

$$|\xi\rangle_{AB} = \sum_{i=1}^r \sigma_i \left(\sum_{j=1}^{m_A} u_{ji} |\psi_j\rangle_A \right) \otimes \left(\sum_{k=1}^{m_B} v_{ik} |\varphi_k\rangle_B \right)$$

$$\bullet \sum_{j=1}^{m_A} u_{ji} |\psi_j\rangle_A = \sum_{j=1}^{m_A} \langle \psi_j, U \varphi_i \rangle \psi_j = U |\varphi_i\rangle_A = |\Psi_i\rangle_A$$

$$\bullet \sum_{k=1}^{m_B} v_{ik} |\psi_k\rangle_B = \sum_{k=1}^{m_B} \langle \psi_i, V \psi_k \rangle |\psi_k\rangle_B = \sum_{k=1}^{m_B} \langle \psi_k, V^\dagger \psi_i \rangle \psi_k = V^\dagger |\psi_i\rangle_B = |\bar{\psi}_i\rangle_B$$

• If we proceeded as the exercise demanded:

$$\langle \bar{\psi}_i | \bar{\psi}_{i'} \rangle = \sum_{j=1}^{m_A} \sum_{j'=1}^{m_A} \underbrace{u_{ji}^* u_{j'i'}}_{\delta_{jj'}} \langle \psi_j | \psi_{j'} \rangle = \sum_{j=1}^{m_A} \underbrace{u_{ji}^* u_{ji}}_{\text{inner product between two columns}} = \delta_{ii'}$$

So $\{|\bar{\psi}_i\rangle\}_{i=1}^n$ is orthonormal, but it need not be a basis! It is only a basis if $n = m_A$ or $n = m_B$. A generic n just satisfies $n \leq \min\{m_A, m_B\}$!

(3.8.2) Let $|\psi\rangle_{AB} \in H_A \otimes H_B$, $\|\psi_{AB}\|_2 = 1$. Assume the Schmidt rank of $|\psi\rangle_{AB}$ is 1. Then, there are states $|\mu\rangle_A$ and $|\nu\rangle_B$ such that

$$|\psi\rangle_{AB} = |\mu\rangle_A \otimes |\nu\rangle_B.$$

This immediately implies that

$$\max_{|\phi\rangle_A, |\eta\rangle_B} |\langle \phi | \otimes \langle \eta | \rangle |\psi\rangle_{AB}|^2 = 1$$

Just pick $|\phi\rangle_A = |\mu\rangle_A$ and $|\eta\rangle_B = |\nu\rangle_B$.

Now, suppose its Schmidt rank is $n > 1$. Then:

$$\begin{aligned} |(\langle \phi |_A \otimes \langle \eta |_B) |\psi\rangle_{AB}|^2 &= \left| \sum_{i=1}^n \sigma_i \langle \phi | \psi_i \rangle_A \langle \eta | \phi_i \rangle_B \right|^2 \\ &\stackrel{\Delta \text{ inequality}}{\leq} \sum_{i=1}^n |\sigma_i \langle \phi | \psi_i \rangle_A \langle \eta | \phi_i \rangle_B|^2 \\ &= \sum_{i=1}^n |\sigma_i|^2 |\langle \phi | \psi_i \rangle_A|^2 |\langle \eta | \phi_i \rangle_B|^2 \\ &\stackrel{|\sigma|^2 = \max_i |\sigma_i|^2}{\leq} |\sigma|^2 \sum_{i=1}^n |\langle \phi | \psi_i \rangle_A|^2 |\langle \eta | \phi_i \rangle_B|^2 \\ &\stackrel{\text{Cauchy Schwarz}}{\leq} |\sigma|^2 \underbrace{\sum_{i=1}^n |\langle \phi | \psi_i \rangle_A|^2}_{=1} \underbrace{\sum_{i=1}^n |\langle \eta | \phi_i \rangle_B|^2}_{=1} \\ &\stackrel{\text{Schmidt rank } > 1 \Rightarrow \sigma_i < 1 \forall i}{<} |\sigma|^2 < 1 \end{aligned}$$

Selected problems from Wilde's book (Chapter 4)

4.1.3

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional Hilbert spaces and suppose that $A \in \text{Lin}(\mathcal{H}_A)$. Also, let $\mathbb{I}_B$ be the identity operator of system B . Also, $\dim \mathcal{H}_A = \dim \mathcal{H}_B = d$

$$\begin{aligned} \text{Tr}(A) &= \text{Tr}(A) = \sum_{i=0}^{d-1} \langle i|A|i \rangle = \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \langle j|A|i \rangle \delta_{ij} \\ &= \sum_{i,j=0}^{d-1} \langle j|A|i \rangle_A \langle j|\mathbb{I}_B|i \rangle_B \\ &= \sum_{i,j=0}^{d-1} \langle j \otimes j, A \otimes \mathbb{I}_B i \otimes i \rangle_{AB} \\ &= \langle \Gamma, A \otimes \mathbb{I}_B \Gamma \rangle_{AB}. \end{aligned}$$

4.1.4

Let $f: S \rightarrow \mathbb{C}$ be a complex-valued function defined

in some appropriate set $S \subseteq \mathbb{C}$ (for instance, S could be a compact topological space). Given $H \in \text{Her}(\mathcal{H}_A)$ with eigenvalues $\{h_1, \dots, h_n\}$, we define the linear operator $f(H)$ as

$$f(H) = \sum_{i: h_i \in S} f(h_i) P_i$$

where P_i is the orthogonal projection onto the i -th eigenspace.

Now, let $G \in \text{Lin}(\mathcal{H}_A)$. The operators $G^\dagger G$ and $GG^\dagger$ are positive, hence, Hermitian. Using the SVD on G , i.e.

$$G = V \Sigma U^t,$$

We obtain

$$\begin{aligned} GG^t &= V \Sigma \Sigma^t V^t = V \Sigma^2 V^t, \\ G^t G &= U \Sigma \Sigma^t U^t = U \Sigma^2 U^t, \end{aligned} \quad (*)$$

because Σ is positive semidefinite when $G \in \text{Lin}(\mathcal{H}_A)$.

But (*) is just the spectral theorem and from it we see that GG^t and $G^t G$ have the same spectrum, i.e. same eigenvalues. Let $\{P_i\}_{i \in \sigma(G^t G)}$ and $\{Q_i\}_{i \in \sigma(GG^t)}$ be the orthogonal projections of $G^t G$ and GG^t , respectively. Then

$$f(G^t G) = \sum_{i: \lambda \in \sigma} f(\lambda_i) P_i$$

$$f(GG^t) = \sum_{i: \lambda \in \sigma} f(\lambda_i) Q_i$$

$$\Rightarrow \left\{ \text{Tr}(f(G^t G)) = \text{Tr}(f(GG^t)) \right\}$$

4.1.9

A state is pure $\Leftrightarrow \text{Tr}(\rho^2) = 1$.

Sufficiency ($\Rightarrow$): suppose ρ is pure. Then $\rho = |\psi\rangle\langle\psi|$ for some $|\psi\rangle \in \mathcal{H}$, $\langle\psi, \psi\rangle = 1$. Then

$$\rho^2 = |\psi\rangle\langle\psi| |\psi\rangle\langle\psi| = \overbrace{\langle\psi|\psi\rangle}^1 |\psi\rangle\langle\psi| = \rho.$$

Thus: $\text{Tr}(\rho^2) = \text{Tr}(\rho) = 1$,
as desired.

Necessity ($\Leftarrow$): let ρ satisfy $\text{Tr}(\rho^2) = 1$.

Given that $\rho \in \text{Pos}(\mathcal{H}) \Leftrightarrow \rho \in \text{Her}(\mathcal{H})$, we can use the spectral theorem and write

$$\rho = \sum_{i=1}^d p_i |\psi_i\rangle\langle\psi_i| \quad ; \quad \langle\psi_i|\psi_j\rangle = \delta_{ij}$$

Where $d = \dim \mathcal{H}$ and $p_1, \dots, p_d$ are non-negative, not necessarily all different. Using this, we have:

$$\begin{aligned} \rho^2 &= \sum_{i=1}^d \sum_{j=1}^d p_i p_j |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j| \\ &= \sum_{i=1}^d p_i^2 |\psi_i\rangle\langle\psi_i|. \end{aligned}$$

Using the hypothesis:

$$\text{Tr}(\rho^2) > 1 \Rightarrow p_1^2 + p_2^2 + \dots + p_d^2 = 1. \quad (1)$$

However, since ρ has unit trace, we also have

$$p_1 + \dots + p_d = 1. \quad (2)$$

Without loss of generality, we can label the coefficients p_i such that the first $n \leq d$ are non-zero. Thus (1) and (2) reduce to

$$\begin{cases} p_1^2 + \dots + p_n^2 = 1 \\ p_1 + \dots + p_n = 1 \end{cases}$$

now with all coefficients strictly greater than 0. Now, assume that $n > 1$. Then for $i = 1, \dots, n$:

$$p_i^2 < p_i < 1,$$

which immediately implies the following inequality:

$$p_1^2 + \dots + p_n^2 < p_1 + \dots + p_n = 1,$$

a clear contradiction with (1). So either $\pi=0$ or $\pi=1$. The option $\pi=0$ leads to ρ being the zero operator, which has trace 0, so $\pi \neq 0$. Thence $\pi=1$, meaning that

$$\rho = |\psi_1\rangle\langle\psi_1|,$$

that is, ρ is a pure state, as desired. $\blacksquare$

4.1.12 A density operator of a qubit system can be represented as

$$\rho = \frac{1}{2} \begin{pmatrix} 1 + \pi_z & \pi_x - i\pi_y \\ \pi_x + i\pi_y & 1 - \pi_z \end{pmatrix}.$$

Let us compute its eigenvalues. The characteristic polynomial is given by

$$\begin{aligned} p_\rho(\lambda) &= \det(\rho - \lambda \mathbb{I}_2) \\ &= \det \begin{pmatrix} \frac{1+\pi_z}{2} - \lambda & \frac{\pi_x - i\pi_y}{2} \\ \frac{\pi_x + i\pi_y}{2} & \frac{1-\pi_z}{2} - \lambda \end{pmatrix} \\ &= \lambda^2 - \left(\frac{1+\pi_z}{2}\right)\lambda - \left(\frac{1-\pi_z}{2}\right)\lambda + \frac{1-\pi_z^2}{4} - \frac{\pi_x^2 - \pi_y^2}{4} \\ &= \lambda^2 - \lambda + \frac{1 - \|\vec{\pi}\|_2^2}{4} \end{aligned}$$

Its roots are: $\lambda_{\pm} = \frac{1 \pm \sqrt{1 - (1 - \|\vec{\pi}\|_2^2)}}{2} = \frac{1 \pm \|\vec{\pi}\|_2}{2}$

4.1.13

Let $\rho = \sum_{j \in J} p(j) |\psi_j\rangle \langle \psi_j|$ and for each $j \in J$:

$$|\psi_j\rangle \langle \psi_j| = \frac{1}{2} (\mathbb{I}_2 + \vec{r}_j \cdot \vec{\sigma}).$$

We have

$$\begin{aligned} \sum_j p(j) |\psi_j\rangle \langle \psi_j| &= \sum_{j \in J} p(j) \left[\frac{1}{2} (\mathbb{I}_2 + \vec{r}_j \cdot \vec{\sigma}) \right] \\ &= \frac{1}{2} \sum_{j \in J} p(j) \mathbb{I}_2 + \frac{1}{2} \sum_{j \in J} p(j) (\vec{r}_j \cdot \vec{\sigma}) \\ &= \frac{1}{2} \mathbb{I}_2 + \frac{1}{2} \left(\sum_{j \in J} p(j) \vec{r}_j \right) \cdot \vec{\sigma} \quad (*) \end{aligned}$$

Now, we know that $\rho = \frac{1}{2} (\mathbb{I}_2 + \vec{r} \cdot \vec{\sigma})$, where $r_i = \text{Tr}(\rho \sigma_i)$ and these coefficients uniquely determine ρ , since $\{\mathbb{I}_2, \sigma_1, \sigma_2, \sigma_3\}$ is a basis. Eq. (*) then tells us that

$$r_i = \sum_{j \in J} p(j) r_j^{(i)}, \quad i = 1, 2, 3.$$

4.1.15

Let $\{|x\rangle\}_{x \in X}$ be an orthonormal basis in $\mathcal{H}$.

What's more, let A and B be non-empty subsets of X and define

$$\Pi(A) = \sum_{x \in A} |x\rangle \langle x| \quad \text{and} \quad \Pi(B) = \sum_{x \in B} |x\rangle \langle x|.$$

Now, suppose $\rho = \sum_{x \in X} p(x) |x\rangle \langle x|$. Then

$$\begin{aligned} \Pi(A) \Pi(B) \rho &= \sum_{x \in A} \sum_{x' \in B} \sum_{x'' \in X} p(x'') \overbrace{|x\rangle \langle x|}^{\delta_{xx'}} \overbrace{|x'\rangle \langle x'|}^{\delta_{x'x''}} |x''\rangle \langle x''| \\ &= \sum_{x \in A \cap B} \sum_{x'' \in X} p(x'') \langle x | x'' \rangle |x\rangle \langle x''| \\ &= \sum_{x \in A \cap B} p(x) |x\rangle \langle x| \quad \rightarrow \text{Tr}(\Pi(A) \Pi(B) \rho) = \sum_{x \in A \cap B} p(x) = p(A \cap B) \end{aligned}$$

4.1.16

Let Π_1, Π_2 be commuting projections and let ρ be an arbitrary state. We will show that

$$\tau_\Pi[(\Pi - \Pi_1 \Pi_2)\rho] \leq \tau_\Pi[(\Pi - \Pi_1)\rho] + \tau_\Pi[(\Pi - \Pi_2)\rho].$$

(I) $\Pi_1 \Pi_2 = \Pi_{12}$ is in $[0, \Pi]$: it suffices to show that it is idempotent and self-adjoint. This makes it an orthogonal projection, so its eigenvalues are 0 and 1.

$$\cdot (\Pi_{12})^2 = (\Pi_1 \Pi_2)(\Pi_1 \Pi_2) = (\Pi_1 \Pi_1)(\Pi_2 \Pi_2) = \Pi_1 \Pi_2$$

$$\cdot \Pi_{12}^\dagger = (\Pi_1 \Pi_2)^\dagger = \Pi_2^\dagger \Pi_1^\dagger = \Pi_2 \Pi_1 = \Pi_1 \Pi_2.$$

$$\begin{aligned} \text{(II)} \quad (\Pi - \Pi_1)(\Pi - \Pi_2) &= \Pi - \Pi_1 - \Pi_2 + \Pi_{12} \\ &= (\Pi - \Pi_1) + (\Pi - \Pi_2) - (\Pi - \Pi_{12}). \end{aligned}$$

Now, for any operator X and positive operator P , we have $X - P \leq X$ (since their diff. is a positive operator).

Take $X = (\Pi - \Pi_1) + (\Pi - \Pi_2)$ and $P = \Pi - \Pi_{12}$. We obtain

$$\Pi - \Pi_{12} \leq (\Pi - \Pi_1) + (\Pi - \Pi_2).$$

Since all involved operators are positive, it holds that for any $\rho \in D(\mathcal{H})$:

$$\tau_\Pi[(\Pi - \Pi_{12})\rho] \leq \tau_\Pi[(\Pi - \Pi_1)\rho] + \tau_\Pi[(\Pi - \Pi_2)\rho],$$

as desired. $\square$

4.2.1

For $k=0, \dots, 4$, define the vectors

$$|e_k\rangle = \cos\left(\frac{2\pi k}{5}\right)|0\rangle + \sin\left(\frac{2\pi k}{5}\right)|1\rangle.$$

We will show that $\left\{\frac{2}{5}|e_k\rangle\langle e_k|\right\}_{k=0}^4$ is a POVM in $\mathbb{C}^2$.

$$\bullet |e_k\rangle\langle e_k| = \begin{pmatrix} \cos^2\left(\frac{2\pi k}{5}\right) & \frac{1}{2}\sin\left(\frac{4\pi k}{5}\right) \\ \frac{1}{2}\sin\left(\frac{4\pi k}{5}\right) & \sin^2\left(\frac{2\pi k}{5}\right) \end{pmatrix}$$

$$\hookrightarrow |e_0\rangle\langle e_0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{aligned} & 2\pi = 2 \cdot \frac{2\pi}{5} + 3 \cdot \frac{2\pi}{5} \quad [x+y=2\pi \Rightarrow \cos x = \cos y] \\ \hookrightarrow & \cos\left(2 \cdot \frac{2\pi}{5}\right) = \cos\left(3 \cdot \frac{2\pi}{5}\right) \end{aligned}$$

$$2\cos^2\left(\frac{2\pi}{5}\right) - 1 = 2\cos^2\left(\frac{2\pi}{5}\right) - 1 + \cos\left(\frac{2\pi}{5} + 2 \cdot \frac{2\pi}{5}\right)$$

$$2\cos^2\left(\frac{2\pi}{5}\right) - 1 = \cos\left(\frac{2\pi}{5}\right)\cos\left(2 \cdot \frac{2\pi}{5}\right) - \sin\left(\frac{2\pi}{5}\right)\sin\left(2 \cdot \frac{2\pi}{5}\right)$$

$$2\cos^2\left(\frac{2\pi}{5}\right) - 1 = 2\cos^3\left(\frac{2\pi}{5}\right) - \cos\left(\frac{2\pi}{5}\right) - 2\sin^2\left(\frac{2\pi}{5}\right)\cos\left(\frac{2\pi}{5}\right)$$

$$2\cos^2\left(\frac{2\pi}{5}\right) - 1 = 2\cos^3\left(\frac{2\pi}{5}\right) - \cos\left(\frac{2\pi}{5}\right) - 2\cos\left(\frac{2\pi}{5}\right) + 2\cos^3\left(\frac{2\pi}{5}\right)$$

$$2\cos^2\left(\frac{2\pi}{5}\right) - 1 = 4\cos^3\left(\frac{2\pi}{5}\right) - 3\cos\left(\frac{2\pi}{5}\right)$$

Thus $\cos\left(\frac{2\pi}{5}\right)$ is one of the roots of the polynomial $q(x) = 4x^3 - 2x^2 - 3x + 1$.

Now

$$4x^3 - 2x^2 - 3x + 1 = 4x^3 - 4x^2 + 2x^2 - 2x - x + 1$$

$$= 4x^2(x-1) + 2x(x-1) - (x-1) = (x-1)(4x^2 + 2x - 1)$$

The roots of $q(x)$ are $1, -\frac{(1+\sqrt{5})}{4}$ and $-\frac{1+\sqrt{5}}{4}$. The only possible choice for $\cos(\frac{2\pi}{5})$ is $-\frac{1+\sqrt{5}}{4}$, since $0 < \frac{2\pi}{5} < \frac{\pi}{2}$ means that $0 < \cos(\frac{2\pi}{5}) < 1$.

Now, let us compute a few cosines and sines.

$$\textcircled{1} \sin^2\left(\frac{2\pi}{5}\right) = 1 - \left(-\frac{1+\sqrt{5}}{4}\right)^2 = 1 - \frac{(\sqrt{5}-1)(\sqrt{5}+1)}{16} = \frac{16-5+2\sqrt{5}-1}{16} = \frac{10+2\sqrt{5}}{16} = \frac{5+\sqrt{5}}{8} = \frac{\sqrt{5}}{8}(1+\sqrt{5})$$

$$\textcircled{2} \cos\left(\frac{2\pi}{5} \cdot 2\right) = 2\cos^2\left(\frac{2\pi}{5}\right) - 1 = 2\left(-\frac{1+\sqrt{5}}{4}\right)^2 - 1 = \frac{6-2\sqrt{5}}{8} - \frac{8}{8} = \frac{-2(1+\sqrt{5})}{8} = \frac{-(1+\sqrt{5})}{4}$$

$$\sin\left(\frac{2\pi}{5} \cdot 2\right) = 2\sin\left(\frac{2\pi}{5}\right)\cos\left(\frac{2\pi}{5}\right) = 2\sqrt{\frac{\sqrt{5}}{8}(\sqrt{5}+1)} \frac{(\sqrt{5}-1)}{4} = \sqrt{\frac{\sqrt{5}}{2}} \frac{\sqrt{\sqrt{5}-1}}{2} = \sqrt{\frac{\sqrt{5}(\sqrt{5}-1)}{8}}$$

$$\textcircled{3} \cos\left(\frac{2\pi}{5} \cdot 3\right) = \cos\left(\frac{2\pi}{5} \cdot 2\right) = \frac{-(1+\sqrt{5})}{4}$$

$$\sin\left(\frac{2\pi}{5} \cdot 3\right) = -\sin\left(\frac{2\pi}{5} \cdot 2\right) = -\sqrt{\frac{\sqrt{5}(\sqrt{5}-1)}{8}}$$

$$\textcircled{4} \cos\left(\frac{2\pi}{5} \cdot 4\right) = 2\cos^2\left(\frac{4\pi}{5}\right) - 1 = 2\cos^2\left(\frac{2\pi}{5} \cdot 2\right) - 1 = \frac{6+2\sqrt{5}}{8} - \frac{8}{8} = \frac{-2+2\sqrt{5}}{8} = \frac{-1+\sqrt{5}}{4}$$

$$\sin\left(\frac{2\pi}{5} \cdot 4\right) = 2\sin\left(\frac{2\pi}{5} \cdot 2\right)\cos\left(\frac{2\pi}{5} \cdot 2\right) = 2\sqrt{\frac{\sqrt{5}(\sqrt{5}-1)}{8}} \frac{-(1+\sqrt{5})}{4} = -\frac{1}{2}\sqrt{\frac{\sqrt{5}(\sqrt{5}-1)}{2}}$$

$$\begin{aligned} \bullet |e_3\rangle\langle e_3| &= \begin{pmatrix} \cos^2\left(\frac{2\pi}{5}\right) & \frac{1}{2}\sin\left(\frac{2\pi}{5} \cdot 2\right) \\ \frac{1}{2}\sin\left(\frac{2\pi}{5} \cdot 2\right) & \sin^2\left(\frac{2\pi}{5}\right) \end{pmatrix} \\ &= \begin{pmatrix} \frac{3-\sqrt{5}}{8} & \frac{1}{2}\sqrt{\frac{\sqrt{5}(\sqrt{5}-1)}{8}} \\ \frac{1}{2}\sqrt{\frac{\sqrt{5}(\sqrt{5}-1)}{8}} & \frac{\sqrt{5}}{8}(1+\sqrt{5}) \end{pmatrix} \end{aligned}$$

$$\bullet |e_2\rangle\langle e_2| = \begin{pmatrix} \cos^2\left(\frac{2\pi}{5} \cdot 2\right) & \sin\left(\frac{2\pi}{5}\right)\cos\left(\frac{2\pi}{5} \cdot 2\right) \\ \sin\left(\frac{2\pi}{5} \cdot 2\right)\cos\left(\frac{2\pi}{5}\right) & \sin^2\left(\frac{2\pi}{5} \cdot 2\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3+\sqrt{5}}{8} & -\frac{1}{2}\sqrt{\frac{\sqrt{5}(\sqrt{5}+1)}{8}} \\ -\frac{1}{2}\sqrt{\frac{\sqrt{5}(\sqrt{5}+1)}{8}} & \frac{\sqrt{5}(\sqrt{5}-1)}{8} \end{pmatrix}$$

$$\bullet |e_3\rangle\langle e_3| = \begin{pmatrix} \cos^2\left(\frac{2\pi}{5} \cdot 3\right) & \sin\left(\frac{2\pi}{5}\right)\cos\left(\frac{2\pi}{5} \cdot 3\right) \\ \sin\left(\frac{2\pi}{5} \cdot 3\right)\cos\left(\frac{2\pi}{5}\right) & \sin^2\left(\frac{2\pi}{5} \cdot 3\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3+\sqrt{5}}{8} & \frac{1}{2}\sqrt{\frac{\sqrt{5}(\sqrt{5}+1)}{8}} \\ \frac{1}{2}\sqrt{\frac{\sqrt{5}(\sqrt{5}+1)}{8}} & \frac{\sqrt{5}(\sqrt{5}-1)}{8} \end{pmatrix}$$

$$\bullet |e_4\rangle\langle e_4| = \begin{pmatrix} \cos^2\left(\frac{2\pi}{5} \cdot 4\right) & \sin\left(\frac{2\pi}{5}\right)\cos\left(\frac{2\pi}{5} \cdot 4\right) \\ \sin\left(\frac{2\pi}{5} \cdot 4\right)\cos\left(\frac{2\pi}{5}\right) & \sin^2\left(\frac{2\pi}{5} \cdot 4\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3-\sqrt{5}}{8} & -\frac{1}{2}\sqrt{\frac{\sqrt{5}(\sqrt{5}-1)}{8}} \\ -\frac{1}{2}\sqrt{\frac{\sqrt{5}(\sqrt{5}-1)}{8}} & \frac{\sqrt{5}(\sqrt{5}+1)}{8} \end{pmatrix}$$

Define $\sum_{k=0}^4 |e_k\rangle\langle e_k| = E$ and let E_{ij} , $i, j \in \{0, 1\}$ be its components.

$$E_{00} = 1 + \frac{3-\sqrt{5}}{8} + \frac{3+\sqrt{5}}{8} + \frac{3+\sqrt{5}}{8} + \frac{3-\sqrt{5}}{8} = \frac{20}{8} = \frac{5}{2}$$

$$E_{01} = E_{10} = 0 + \frac{1}{2} \sqrt{\frac{\sqrt{5}(\sqrt{5}-1)}{8}} - \frac{1}{2} \sqrt{\frac{\sqrt{5}(\sqrt{5}+1)}{8}} + \frac{1}{2} \sqrt{\frac{\sqrt{5}(\sqrt{5}+1)}{8}} - \frac{1}{2} \sqrt{\frac{\sqrt{5}(\sqrt{5}-1)}{8}} = 0$$

$$E_{11} = 0 + \frac{\sqrt{5}(1+\sqrt{5})}{8} + \frac{\sqrt{5}(\sqrt{5}-1)}{8} + \frac{\sqrt{5}(\sqrt{5}-1)}{8} + \frac{\sqrt{5}(\sqrt{5}+1)}{8}$$

$$= \frac{1}{8} [\sqrt{5} + 5 + 5 - \sqrt{5} + 5 - \sqrt{5} + 5 + \sqrt{5}] = \frac{20}{8} = \frac{5}{2}$$

Hence $E = \frac{5}{2} \mathbb{I} \Rightarrow \sum_{k=0}^4 \frac{2}{5} |e_k\rangle\langle e_k| = \mathbb{I}_2$

4.2.2

Upper bound on state discrimination probability

Given an ensemble $\{p(x), \rho_x\}_{x \in X}$ and a POVM $\{M_x\}_{x \in X}$ suppose there is an operator τ s.t. $p(x)\rho_x \leq \tau \quad \forall x \in X$.

(I) τ is positive: $\langle \psi | \tau - p(x)\rho_x | \psi \rangle \geq 0 \Rightarrow \langle \psi | \tau | \psi \rangle \geq p(x) \langle \psi | \rho_x | \psi \rangle \geq 0$.

(II) $\tau - p(x)\rho_x \geq 0 \Rightarrow \forall E \in [0, \mathbb{I}] : \text{Tr}[(\tau - p(x)\rho_x)E] \geq 0$. Hence

$$\forall E \in [0, \mathbb{I}] : \text{Tr}(E\tau) \geq \text{Tr}(E p(x)\rho_x).$$

(III) Since $M_x \in [0, \mathbb{I}]$ for any $x \in X$, we have

$$\sum_{x \in X} p(x) \text{Tr}(M_x \rho_x) = \sum_{x \in X} \text{Tr}(M_x p_x \rho_x) \leq \sum_{x \in X} \text{Tr}(M_x \tau) = \text{Tr}\left(\sum_{x \in X} M_x \tau\right) = \text{Tr}(\tau)$$

as desired.

(IV) Now, consider $X = \{0,1\}^n$ and $p(x) = \frac{1}{2^n}$. We can construct τ using the fact that $p \leq \Pi \quad \forall p \in D(\mathcal{H})$. All we have to do is set $\tau = \frac{1}{2^n} \cdot \Pi$. By the previous result, for any POVM we see that its guessing probability is upper bounded by $\text{Tr}(\tau) = \frac{d}{2^n}$. The only possibility for perfect discrimination occurs when $d = 2^n$, i.e. we encode n classical bits in n qubits (on a system of similar dimension).

4.3.1 Define SWAP: $\mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ as

$$\text{SWAP} |\psi\rangle \otimes |\varphi\rangle = |\varphi\rangle \otimes |\psi\rangle$$

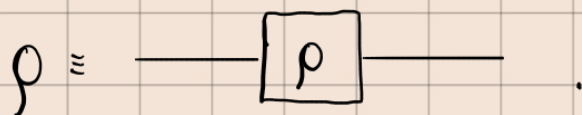
for any $\psi, \varphi \in \mathcal{H}$.

For any $p \in D(\mathcal{H})$, its purity is given by $\text{Tr}(p^2)$. We will show that SWAP is a operation that satisfies

$$\text{Tr}(p^2) = \text{Tr}(p \otimes p \circ \text{SWAP})$$

for $\text{SWAP} \in \text{Lin}(\mathcal{H} \otimes \mathcal{H})$.

For this, it is useful to represent any linear operation as a box with input and output wires:



The trace of an operator is symbolized by joining the tips of the input and output wires:

$$\text{Tr } \rho \equiv \text{[Diagram: A horizontal line with a box labeled } \rho \text{ in the middle, enclosed in a large loop below it.]}$$

In this diagrammatic representation, we want to find O such that

$$\text{[Diagram: Two boxes labeled } \rho \text{ connected by a horizontal line, enclosed in a large loop below it.] } \equiv \text{[Diagram: A box labeled } O \text{ connected to two boxes labeled } \rho \text{, enclosed in a large loop above and below it.]}$$

The SWAP operator acts as follows:

$$\text{[Diagram: Two horizontal lines entering a box labeled SWAP, then exiting to two boxes labeled } \rho \text{.] } \equiv \text{[Diagram: Two horizontal lines crossing each other, then exiting to two boxes labeled } \rho \text{.]}$$

Thus:

$$\text{[Diagram: A box labeled SWAP connected to two boxes labeled } \rho \text{, enclosed in a large loop above and below it.] } \equiv \text{[Diagram: Two boxes labeled } \rho \text{ connected by a horizontal line, enclosed in a large loop below it, with blue and pink arrows indicating a swap operation.]}$$

If you prefer the long and boring calculation:

• Let $|\psi\rangle$ be any pure state, we know that $\text{Tr}(|\psi\rangle\langle\psi|^2) = 1$.

Since

$$\text{SWAP } |\psi\rangle\otimes|\psi\rangle = |\psi\rangle\otimes|\psi\rangle,$$

We also have:

$$\text{Tr}((\text{SWAP } |\psi\rangle\otimes|\psi\rangle)\langle\psi|\langle\psi|) = \text{Tr}(|\psi\rangle\otimes|\psi\rangle\langle\psi|\langle\psi|) = 1$$

• Now let $\rho \in D(\mathcal{H})$. Write it as $\rho = \sum_i \alpha_i |\psi_i\rangle\langle\psi_i|$. Then.

$$\rho \otimes \rho = \sum_{i,j} \alpha_i \alpha_j |\psi_i\rangle\langle\psi_i| \otimes |\psi_j\rangle\langle\psi_j|.$$

Consequently:

$$\begin{aligned} \text{Tr}(\text{SWAP } \rho \otimes \rho) &= \sum_{i,j} \alpha_i \alpha_j \text{Tr}[\text{SWAP}(|\psi_i\rangle\otimes|\psi_j\rangle)\langle\psi_i|\langle\psi_j|] \\ &= \sum_{i,j} \alpha_i \alpha_j \text{Tr}(|\psi_j\rangle\langle\psi_i| \otimes |\psi_i\rangle\langle\psi_j|) \\ &= \sum_{i,j} \alpha_i \alpha_j \overbrace{\text{Tr}(|\psi_j\rangle\langle\psi_i|)}^{\delta_{ji}} \overbrace{\text{Tr}(|\psi_i\rangle\langle\psi_j|)}^{\delta_{ij}} \\ &= \sum_i \alpha_i^2 = \text{Tr}(\rho^2). \end{aligned}$$

If $f: S \rightarrow \mathbb{C}$, $S \subset \mathbb{C}$, let us write $f(\rho) = \sum_{i: \rho_i \in S} f(\rho_i) |\psi_i\rangle\langle\psi_i|$.

Then

$$\begin{aligned} \text{Tr}[f(\rho) \otimes \text{SWAP}] &= \sum_{i: \rho_i \in S} f(\rho_i) \text{Tr}(\text{SWAP } |\psi_i\rangle\langle\psi_i| \otimes \sum_j |\psi_j\rangle\langle\psi_j|) \\ &= \sum_{i: \rho_i \in S} f(\rho_i) \sum_j \overbrace{\text{Tr}(|\psi_i\rangle\langle\psi_j|)}^{\delta_{ij}} \overbrace{\text{Tr}(|\psi_j\rangle\langle\psi_i|)}^{\delta_{ji}} \\ &= \sum_{i: \rho_i \in S} f(\rho_i) = \text{Tr}(f(\rho)). \quad \square \end{aligned}$$

4.3.13 Let X_{AB} be a linear operator on $\mathcal{H}_A \otimes \mathcal{H}_B$.

Easy solution: since $\text{Lin}(\mathcal{H}_A \otimes \mathcal{H}_B) \simeq \text{Lin}(\mathcal{H}_A) \otimes \text{Lin}(\mathcal{H}_B)$, we can
 Better call it basis-independent solution
 always write X_{AB} as a finite linear combination of product operators:

$$X_{AB} = \sum_{i=1}^m X_A^{(i)} \otimes X_B^{(i)}.$$

Consequently:

$$\begin{aligned} \text{Tr}_A X_{AB} &= \sum_{i=1}^m \text{Tr}(X_A^{(i)}) X_B^{(i)}; \\ \text{Tr}_B X_{AB} &= \sum_{i=1}^m \text{Tr}(X_B^{(i)}) X_A^{(i)} \end{aligned}$$

$$\begin{aligned} \text{Hence: } \text{Tr}_B \text{Tr}_A X_{AB} &= \sum_{i=1}^m \text{Tr}(X_A^{(i)}) \text{Tr}(X_B^{(i)}) \\ \text{Tr}_A \text{Tr}_B X_{AB} &= \sum_{i=1}^m \text{Tr}(X_B^{(i)}) \text{Tr}(X_A^{(i)}) \end{aligned}$$

Which indeed are the same.

Lengthy solution: fix ^{arbitrary orthonormal} bases $\{|v_i\rangle\}_{i \in I} \subset \mathcal{H}_A$ and $\{|u_j\rangle\}_{j \in J} \subset \mathcal{H}_B$.
 write down

$$X_{AB} = \sum_{i_1, i_2 \in I} \sum_{j_1, j_2 \in J} \langle v_{i_1} \otimes u_{j_1} | X_{AB} | v_{i_2} \otimes u_{j_2} \rangle \overbrace{|v_{i_1} \otimes u_{j_1}\rangle \langle v_{i_2} \otimes u_{j_2}|}^{|v_{i_1}\rangle \langle v_{i_2}| \otimes |u_{j_1}\rangle \langle u_{j_2}|}$$

Use the fact that $\text{Tr}(|v_{i_1}\rangle \langle v_{i_2}|) = \delta_{i_1, i_2}$ and $\text{Tr}(|u_{j_1}\rangle \langle u_{j_2}|) = \delta_{j_1, j_2}$ so
 verify the claim.

4.3.14

Let $U_B \in U(H_B)$ and fix $\rho_{AB} \in D(H_A \otimes H_B)$. Write $\rho_{AB} = \sum_{i=1}^{\infty} \chi_A^{(i)} \otimes \chi_B^{(i)}$.

Then:

$$\begin{aligned} \sigma_A &\doteq \text{Tr}_B[(\mathbb{I}_A \otimes U_B) \rho_{AB} (\mathbb{I}_A \otimes U_B^\dagger)] = \text{Tr}_B\left[\sum_{i=1}^{\infty} \chi_A^{(i)} \otimes U_B \chi_B^{(i)} U_B^\dagger\right] \\ &= \sum_{i=1}^{\infty} \text{Tr}(U_B \chi_B^{(i)} U_B^\dagger) \chi_A^{(i)} \\ &= \sum_{i=1}^{\infty} \text{Tr}(U_B^\dagger U_B \chi_B^{(i)}) \chi_A^{(i)} \\ &= \sum_{i=1}^{\infty} \text{Tr}(\chi_B^{(i)}) \chi_A^{(i)} = \text{Tr}_B \rho_{AB} = \rho_A. \end{aligned}$$

Conclusion: local unitaries on Bob's side don't affect ρ_A .

Now, let us consider a local measurement $\{M_j\}_{j \in J} \subset \text{Lin}(H_B)$. For each outcome j , the global state is:

$$\begin{aligned} \sigma_{AB}^{(j)} &= \frac{(\mathbb{I}_A \otimes M_j) \rho_{AB} (\mathbb{I}_A \otimes M_j^\dagger)}{\text{Tr}[(\mathbb{I}_A \otimes M_j) \rho_{AB} (\mathbb{I}_A \otimes M_j^\dagger)]} \\ &= \frac{(\mathbb{I}_A \otimes M_j) \rho_{AB} (\mathbb{I}_A \otimes M_j^\dagger)}{\text{Tr}(M_j^\dagger M_j \rho^B)}. \end{aligned}$$

So, upon occurrence of outcome j , Alice's system is

$$\sigma_A^{(j)} = \frac{\text{Tr}_B[(\mathbb{I}_A \otimes M_j) \rho_{AB} (\mathbb{I}_A \otimes M_j^\dagger)]}{\text{Tr}(M_j^\dagger M_j \rho^B)}.$$

If Bob does not communicate which outcome he observed, Alice's system is described by the ensemble $\{(\text{Tr}(M_j^\dagger M_j \rho^B), \sigma_A^{(j)})\}_{j \in J}$.

Hence, the density operation is

$$\sigma_A = \sum_{j \in J} \text{Tr}(M_j^\dagger M_j \rho^B) \sigma_A^{(j)} = \sum_{j \in J} \text{Tr}_B[(\mathbb{I}_A \otimes M_j) \rho_{AB} (\mathbb{I}_A \otimes M_j^\dagger)]$$

Once again, write $\rho_{AB} = \sum_{i=1}^m X_A^{(i)} \otimes X_B^{(i)}$. This leads to

$$\begin{aligned} \sigma_A &= \sum_{j \in J} \sum_{i=1}^m \text{Tr}(M_j^\dagger M_j X_B^{(i)}) X_A^{(i)} = \sum_{i=1}^m \text{Tr} \left[\sum_{j \in J} \overbrace{M_j^\dagger M_j}^{\mathbb{I}} X_B^{(i)} \right] X_A^{(i)} \\ &= \sum_{i=1}^m \text{Tr}(X_B^{(i)}) X_A^{(i)} = \rho_A. \end{aligned}$$

4.3.15 Let $X_{AB} \in \text{Lim}(H_A \otimes H_B)$, $W_B, Y_B, Z_B \in \text{Lim}(H_B)$.

$$\begin{aligned} \text{Claim: } \text{Tr}_B [X_{AB} (\mathbb{I}_A \otimes Y_B Z_B W_B)] &= \text{Tr}_B [(\mathbb{I}_A \otimes W_B) X_{AB} (\mathbb{I}_A \otimes Y_B Z_B)] \\ &= \text{Tr}_B [(\mathbb{I}_A \otimes Z_B W_B) X_{AB} (\mathbb{I}_A \otimes Y_B)] \\ &= \text{Tr}_B [(\mathbb{I}_A \otimes Y_B Z_B W_B) X_{AB}]. \end{aligned}$$

Proof. Again, write $X_{AB} = \sum_{i=1}^m X_A^{(i)} \otimes X_B^{(i)}$. Then:

$$\begin{aligned} \text{Tr}_B [X_{AB} (\mathbb{I}_A \otimes Y_B Z_B W_B)] &= \text{Tr}_B \left[\sum_{i=1}^m X_A^{(i)} \otimes X_B^{(i)} Y_B^{(i)} Z_B^{(i)} W_B^{(i)} \right] \\ &= \sum_{i=1}^m \text{Tr}_B (X_B^{(i)} Y_B Z_B W_B) X_A^{(i)}. \end{aligned}$$

By cyclicity of the trace: $\text{Tr}(X_B^{(i)} Y_B Z_B W_B) = \text{Tr}(W_B X_B^{(i)} Y_B Z_B)$.

As consequence:

$$\text{Tr}_B (X_B^{(i)} Y_B Z_B W_B) X_A^{(i)} = \text{Tr}_B (W_B X_B^{(i)} Y_B Z_B) X_A^{(i)} = \text{Tr}_B [X_A^{(i)} \otimes W_B X_B^{(i)} Y_B Z_B].$$

Using $AB \otimes CD = (A \otimes C)(B \otimes D)$ twice, we can write

$$X_A^{(i)} \otimes W_B X_B^{(i)} Y_B Z_B = (\mathbb{I}_A \otimes W_B) (X_A^{(i)} \otimes X_B^{(i)}) (\mathbb{I}_A \otimes Y_B Z_B)$$

Hence
$$T_{\Pi}(X_B^{(i)} Y_B Z_B W_B) X_A^{(i)} = T_{\Pi_B}[(\mathbb{I}_A \otimes W_B)(X_A^{(i)} \otimes X_B^{(i)})(\mathbb{I}_A \otimes Y_B Z_B)]$$

and
$$\sum_{i=1}^m T_{\Pi}(X_B^{(i)} Y_B Z_B W_B) X_A^{(i)} = T_{\Pi_B}[(\mathbb{I}_A \otimes W_B)(\sum_{i=1}^m X_A^{(i)} \otimes X_B^{(i)})(\mathbb{I}_A \otimes Y_B Z_B)]$$

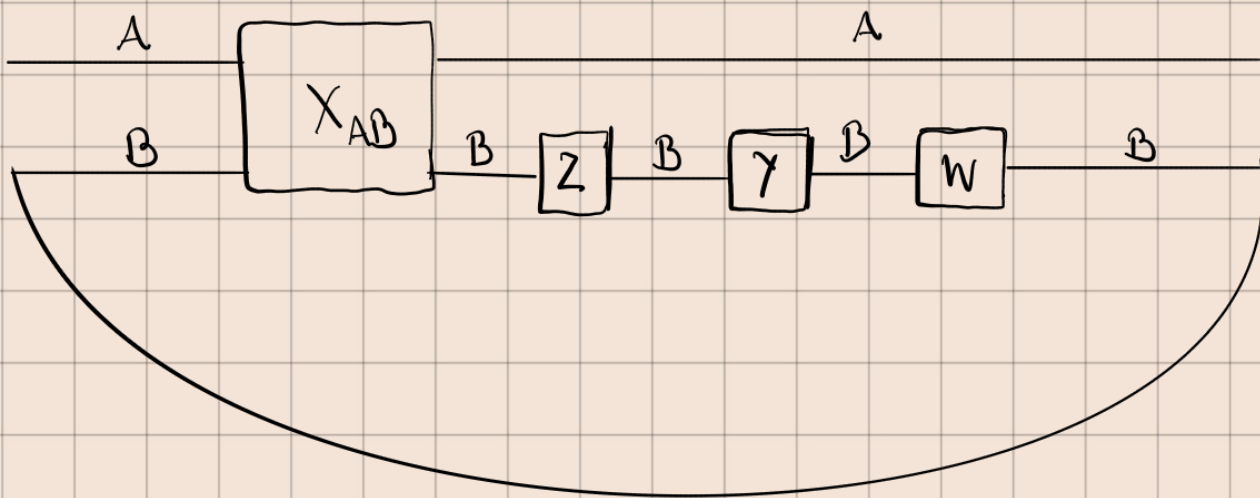
$$= T_{\Pi_B}[(\mathbb{I}_A \otimes W_B) \chi_{AB} (\mathbb{I}_A \otimes Y_B Z_B)]$$

Conclusion

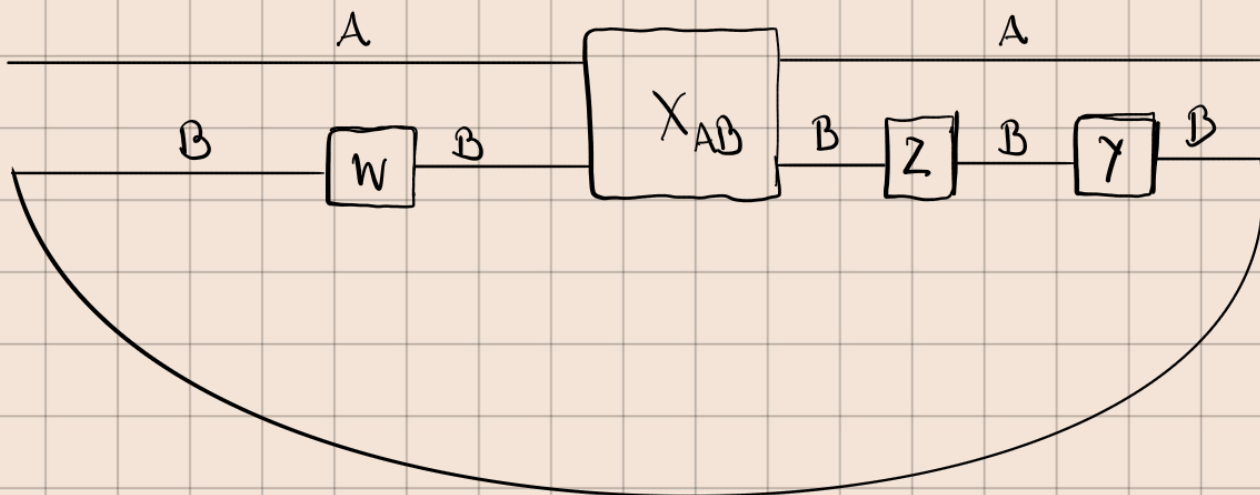
$$T_{\Pi_B}[\chi_{AB} (\mathbb{I}_A \otimes Y_B Z_B W_B)] = T_{\Pi_B}[(\mathbb{I}_A \otimes W_B) \chi_{AB} (\mathbb{I}_A \otimes Y_B Z_B)]$$

The remaining equalities are proved similarly. $\square$

Diagrammatically:



|||



4.3.16

The maximally entangled state $|\Phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle$ has as reduced state

$$\rho_A = \frac{1}{d} \sum_{i=0}^{d-1} |i\rangle\langle i| = \frac{1}{d} \mathbb{I}_d = \pi_d. \quad (\text{Maximally mixed})$$

From this, we have $\rho_A^2 = \frac{1}{d^2} \mathbb{I}_d \Rightarrow \boxed{\text{Tr}(\rho_A^2) = 1/d}$, as desired. $\square$

4.3.19

Let $D_{cq}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be the set of classical-quantum states, i.e. states of the form

$$\rho^{cq} = \sum_x p_x(x) |x\rangle\langle x|_A \otimes \rho_B^x$$

Where X is some RV with dist. $p_x(x)$, $\{|x\rangle\}_x \subset \mathcal{H}_A$ is an orthonormal basis, and $\rho_B^x \in D(\mathcal{H}_B)$.

(I) Prove that if ρ^{cq} and σ^{cq} are classical-quantum w.r.t. the same basis, then $\lambda \rho^{cq} + (1-\lambda) \sigma^{cq}$ is classical-quantum.

(II) Provide a counter-example to show $D_{cq}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is not convex. The states must necessarily be cq in distinct bases!

The Choi-Kraus Theorem

A linear map $L: \text{Lin}(\mathcal{H}_A) \rightarrow \text{Lin}(\mathcal{H}_B)$ is completely positive and trace-preserving if and only if there exist linear operators $V_i: \mathcal{H}_A \rightarrow \mathcal{H}_B$, $i=1, \dots, d$ such that $\forall X \in \text{Lin}(\mathcal{H}_A)$:

$$L(X) = \sum_{i=1}^d V_i X V_i^\dagger \quad (*)$$

where $d \leq \min\{d_A, d_B\}$, and

$$\sum_{i=1}^d V_i^\dagger V_i = \mathbb{I}_A. \quad (**)$$

Proof. Assume that (*) and (**) hold, and let H_m be an m -dimensional Hilbert space. Let $X \in \text{Lin}(H_m \otimes H_A)$. Since $\text{Lin}(H_m \otimes H_A)$ is spanned by product operators, we can decompose X as

$$X = \sum_{i=1}^m Y_i \otimes Z_i,$$

where $Y_i \in \text{Lin}(H_m)$ and $Z_i \in \text{Lin}(H_A) \forall i$.

Consequently, the action of $\text{id}_m \otimes L: \text{Lin}(H_m \otimes H_A) \rightarrow \text{Lin}(H_m \otimes H_B)$ on X is:

$$\begin{aligned} (\text{id}_m \otimes L)(X) &= \sum_{i=1}^m \text{id}_m(Y_i) \otimes L(Z_i) \\ &= \sum_{i=1}^m Y_i \otimes \left(\sum_{j=1}^d V_j Z_i V_j^\dagger \right) \\ &= \sum_{i=1}^m \sum_{j=1}^d Y_i \otimes V_j Z_i V_j^\dagger \end{aligned}$$

Now, assume that X is a positive operator and let $v \in H_m \otimes H_B$. Then:

$$\langle v, (\text{id}_m \otimes L)(X) v \rangle = \sum_{i=1}^m \sum_{j=1}^d \langle v, (Y_i \otimes V_j Z_i V_j^\dagger) v \rangle.$$

Notice that

$$Y_i \otimes V_j Z_i V_j^\dagger = (\mathbb{I}_m \otimes V_j)(Y_i \otimes Z_i)(\mathbb{I}_m \otimes V_j^\dagger).$$

Using the previous equality:

$$\begin{aligned}
 \langle v, (Y_i \otimes V_j Z_i V_j^\dagger) v \rangle &= \langle v, (\mathbb{I}_m \otimes V_j)(Y_i \otimes Z_i)(\mathbb{I}_m \otimes V_j^\dagger) v \rangle \\
 &= \langle (\mathbb{I}_m \otimes V_j^\dagger) v, (Y_i \otimes Z_i)(\mathbb{I}_m \otimes V_j) v \rangle \\
 &= \langle w_j, (Y_i \otimes Z_i) w_j \rangle
 \end{aligned}$$

where $w_j = (\mathbb{I}_m \otimes V_j^\dagger) v$. Consequently:

$$\begin{aligned}
 \langle v, (\text{id}_m \otimes L)(X) v \rangle &= \sum_{i=1}^m \sum_{j=1}^d \langle w_j, (Y_i \otimes Z_i) w_j \rangle \\
 &= \sum_{j=1}^d \langle w_j, \left(\sum_{i=1}^m Y_i \otimes Z_i \right) w_j \rangle \\
 &= \sum_{j=1}^d \langle w_j, X w_j \rangle \geq 0. \quad (***)
 \end{aligned}$$

Since the dimension of $\mathcal{H}_m$ is arbitrary, (***) holds for any $m \in \mathbb{N}$. Hence, L is completely-positive. Checking that it is trace-preserving is trivial.

Now, we prove the opposite direction. Let $\mathcal{H}_{A'}$ be another Hilbert space with $\dim \mathcal{H}_{A'} = \dim \mathcal{H}_A = d_A$. In the canonical basis $\{|i\rangle\}_{i=1}^{d_A}$, define

$$|\Gamma\rangle_{AA'} \equiv \sum_{i=1}^{d_A} |i\rangle_A \otimes |i\rangle_{A'}.$$

We observe three things:

- 1) Since $|\Gamma\rangle\langle\Gamma|_{AA} \geq 0$ and L is CPTP, we have $\text{id}_{A'} \otimes L(|\Gamma\rangle\langle\Gamma|_{AA}) \geq 0$. Hence, due to the spectral thm:

$$J(L) \equiv \text{id}_{A'} \otimes L(|\Gamma\rangle\langle\Gamma|_{AA}) = \sum_{l=1}^n |\Phi_l\rangle\langle\Phi_l|_{A'A}$$

where $\pi \equiv \text{rank}(J(L))$ and $|\phi\rangle$ not necessarily normalised.

2) For any vector $v \in \mathcal{H}_A \otimes \mathcal{H}_B$, there is a linear map $V: \mathcal{H}_A \rightarrow \mathcal{H}_B$ such that

$$(\mathbb{I}_A \otimes V) |\Gamma_{AA}\rangle = |v\rangle.$$

To show this, use the comp. basis in $\mathcal{H}_A$ and $\mathcal{H}_B$ to write

$$v = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \alpha_{ij} |i\rangle_A \otimes |j\rangle_B$$

$$V = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \alpha_{ij} |j\rangle_B \langle i|_A.$$

3) For any i, j in $\{1, \dots, d_A\}$:

$$\begin{aligned} \text{Tr}_A(J(L) |j\rangle\langle i|_A \otimes \mathbb{I}_B) &= \text{Tr}_A((|j\rangle\langle i|_A \otimes L)(\Gamma \chi \Gamma^\dagger)) \\ &= L(|i\rangle\langle j|_A). \end{aligned}$$

From 3, 1 and 2:

$$\begin{aligned} L(|i\rangle\langle j|_A) &= \text{Tr}_A[J(L) |j\rangle\langle i| \otimes \mathbb{I}_B] \\ &= \text{Tr}_A\left[\left(\sum_{\ell=1}^{\pi} |\phi_\ell\rangle\langle\phi_\ell|_{AB}\right) |j\rangle\langle i|_A \otimes \mathbb{I}_B\right] \\ &= \text{Tr}_A\left[\sum_{\ell=1}^{\pi} \left((\mathbb{I}_A \otimes V_\ell) \Gamma \chi \Gamma_{AA}^\dagger (\mathbb{I}_A \otimes V_\ell^\dagger) |j\rangle\langle i|_A \otimes \mathbb{I}_B\right)\right] \\ &= \sum_{\ell=1}^{\pi} \text{Tr}_A\left((\mathbb{I}_A \otimes V_\ell) \Gamma \chi \Gamma_{AA}^\dagger (\mathbb{I}_A \otimes V_\ell^\dagger) |j\rangle\langle i|_A \otimes \mathbb{I}_B\right) \\ &= \sum_{\ell=1}^{\pi} V_\ell |i\rangle\langle j|_A V_\ell^\dagger \end{aligned}$$

Which is the desired decomposition. Lastly:

$$\begin{aligned}
\sum_{\ell=1}^n V_{\ell}^{\dagger} V_{\ell} &= \sum_{\ell=1}^n \left(\sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \alpha_{ij}^{\ell*} |i\rangle\langle j|_A \right) \left(\sum_{i'=1}^{d_A} \sum_{j'=1}^{d_B} \alpha_{i'j'}^{\ell} |j'\rangle\langle i'|_A \right) \\
&= \sum_{\ell=1}^n \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \sum_{i'=1}^{d_A} (\alpha_{ij}^{\ell})^* \alpha_{i'j}^{\ell} |i\rangle\langle i'|_A \\
&= \sum_{\ell=1}^n T_{\Pi_B} (|\phi_{\ell}\rangle\langle\phi_{\ell}|) \\
&= T_{\Pi_B} \left(\sum_{\ell=1}^n |\phi_{\ell}\rangle\langle\phi_{\ell}| \right) \\
&= T_{\Pi_B} (J(L)) = \sum_{i=1}^{d_A} \sum_{j=1}^{d_A} |i\rangle\langle j|_A \underbrace{T_{\Pi} (L(|i\rangle\langle j|_A))}_{= T_{\Pi} (|i\rangle\langle j|) = \delta_{ij}} \\
&= \sum_{i=1}^{d_A} |i\rangle\langle i|_A = \Pi_{A'} = \Pi_A. \quad \square
\end{aligned}$$

4.4.1

Let $L: \text{Lin}(\mathcal{H}_A) \rightarrow \text{Lin}(\mathcal{H}_B)$ be such that $J(L) \geq 0$. Then L is completely positive.

Proof.

$$J(L) = (\text{id}_A \otimes L) |\Gamma\rangle\langle\Gamma|_{AA} = \sum_{n=1}^n |\phi_n\rangle\langle\phi_n|_{AB}.$$

$$1) J(L) \geq 0 \Rightarrow L \geq 0.$$

By direct computation:

$$L(X) = T_{\Pi_{A'}} (J(L)(X^T \otimes \Pi_B)).$$

$$\langle u | L(X) | v \rangle_B = T_{\Pi_B} T_{\Pi_{A'}} (J(L)(X^T \otimes |u\rangle\langle v|))$$

$$= T_{\Pi} \left(\underbrace{J(L)}_{\geq 0} \left(\underbrace{X^T \otimes |u\rangle\langle v|}_{\geq 0} \right) \right) \geq 0 \quad \because L(X) \geq 0$$

2) Is L 2-positive? Let $\{|e_i\rangle_A\}_{i=0}^{d_A-1}$ be the canonical basis of $\mathcal{H}_A$. Any vector $|x\rangle \in \mathbb{C}^2 \otimes \mathcal{H}_A$ can be written as

$$|x\rangle = |0\rangle_{\mathbb{C}^2} \otimes |u_0\rangle_A + |1\rangle_{\mathbb{C}^2} \otimes |u_1\rangle_A \quad (\Delta)$$

To see this, just write $|x\rangle = \sum_{i=0}^1 \sum_{j=0}^{d_A-1} \alpha_{ij} |i\rangle_{\mathbb{C}^2} \otimes |e_j\rangle_A$ and set $|u_i\rangle \equiv \sum_{j=0}^{d_A-1} \alpha_{ij} |e_j\rangle_A$.

Using (Δ) , the action of $\text{id}_2 \otimes L$ on $|x\rangle\langle x|$ is

$$(\text{id}_2 \otimes L)(|x\rangle\langle x|) = \sum_{i=0}^1 \sum_{j=0}^1 |i\rangle\langle j|_{\mathbb{C}^2} \otimes L(|u_i\rangle\langle u_j|) \quad (\Delta\Delta)$$

Hence, for a vector $|y\rangle = |0\rangle_{\mathbb{C}^2} \otimes |v_0\rangle_B + |1\rangle_{\mathbb{C}^2} \otimes |v_1\rangle_B$ in $\mathbb{C}^2 \otimes \mathcal{H}_B$:

$$\begin{aligned} \langle y | (\text{id}_2 \otimes L)(|x\rangle\langle x|) | y \rangle &= \sum_{i,j=0}^1 \langle y | (|i\rangle\langle j|_{\mathbb{C}^2} \otimes L(|u_i\rangle\langle u_j|)) | y \rangle \\ &= \sum_{i,j=0}^1 \langle v_i | L(|u_i\rangle\langle u_j|) | v_j \rangle. \end{aligned}$$

Now, write $L(|u_i\rangle\langle u_j|) = \sum_{k=0}^{d_A-1} \sum_{l=0}^{d_A-1} \alpha_{ik} \alpha_{jl}^* L(|e_k\rangle\langle e_l|)$. We have

$$\begin{aligned} \langle y | (\text{id}_2 \otimes L)(|x\rangle\langle x|) | y \rangle &= \sum_{i,j=0}^1 \sum_{k,l=0}^{d_A-1} \alpha_{ik} \alpha_{jl}^* \langle v_i | L(|e_k\rangle\langle e_l|) | v_j \rangle \\ &= \sum_{i,j=0}^1 \sum_{k,l=0}^{d_A-1} \sum_{m,n=0}^{d_B-1} \alpha_{ik} \alpha_{jl}^* \beta_{im}^* \beta_{jn} \langle f_m | L(|e_k\rangle\langle e_l|) | f_n \rangle \\ &= \sum_{k,l=0}^{d_A-1} \sum_{m,n=0}^{d_B-1} \langle f_m | L(|e_k\rangle\langle e_l|) | f_n \rangle \left(\sum_{i,j=0}^1 \alpha_{ik} \beta_{im}^* \alpha_{jl}^* \beta_{jn} \right) \end{aligned}$$

Set $|z\rangle \equiv \sum_{k=0}^{d_A-1} \sum_{m=0}^{d_B-1} \underbrace{\left(\sum_{i=0}^1 \alpha_{ik}^* \beta_{im} \right)}_{\equiv \gamma_{km}} |e_k\rangle \otimes |f_m\rangle$. Then:

$$\begin{aligned}
\langle y | (id_2 \otimes L)(|x\rangle\langle x|) | y \rangle &= \sum_{k,l=0}^{d_A-1} \sum_{m,n=0}^{d_B-1} \langle f_m | L(|e_k\rangle\langle e_l|) | f_m \rangle \gamma_{km}^* \gamma_{ln} \\
&= \sum_{k,l=0}^{d_A-1} \sum_{m,n=0}^{d_B-1} \sum_{p,q=0}^{d_A-1} \langle f_m | L(|e_k\rangle\langle e_l|) | f_m \rangle \gamma_{pm}^* \gamma_{qn} \langle e_p | e_k \rangle \langle e_l | e_q \rangle \\
&= \left(\sum_{p=0}^{d_A-1} \sum_{m=0}^{d_B-1} \gamma_{pm}^* \langle e_p \otimes f_m | \right) \left(\sum_{k,l=0}^{d_A-1} |e_k\rangle\langle e_l| \otimes L(|e_k\rangle\langle e_l|) \right) \left(\sum_{q=0}^{d_A-1} \sum_{n=0}^{d_B-1} \gamma_{qn} |e_q \otimes f_n\rangle \right) \\
&= \langle z | J(L) | z \rangle.
\end{aligned}$$

Since $J(L) \geq 0$, it follows that $\langle z | J(L) | z \rangle$ is equal to or greater than zero. As consequence:

$$\langle y | (id_2 \otimes L)(|x\rangle\langle x|) | y \rangle \geq 0$$

for fixed $|x\rangle$ and arbitrary $|y\rangle$. Thus $(id_2 \otimes L)(|x\rangle\langle x|)$ is a positive operator.

Lastly, let $X \in L_{\text{im}}(\mathbb{C}^2 \otimes \mathcal{H}_A)$ be positive. Then, we can write

$$X = \sum_{\ell=1}^n |x_\ell\rangle\langle x_\ell|$$

where $n = \text{rank}(X)$. Due to linearity:

$$(id_2 \otimes L)(X) = \sum_{\ell=1}^n (id_2 \otimes L)(|x_\ell\rangle\langle x_\ell|).$$

From the previous steps, we have that each term in sum on the RHS is a positive operator. This by itself implies that the operator on the RHS, and consequently $(id_2 \otimes L)(X)$, is positive. From this, we see that

L is 2-positive!

We can replace 2 by an arbitrary dimension $n \in \mathbb{N}$, $n \geq 2$, w.l.o.g. Thus L is n -positive

for any n , which means that L is completely positive. $\square$

4.6.19

A channel $L_{cq} : \text{Lin}(\mathcal{H}_A) \rightarrow \text{Lin}(\mathcal{H}_B)$ is classical-quantum if there's an orthonormal basis $\{|\varphi_i\rangle\}_{i=0}^{d_A-1} \subset \mathcal{H}_A$ and states $\{\sigma_{i,B}\}_{i=0}^{d_A-1}$ such that

$$\forall P \in \text{Lin}(\mathcal{H}_A) : L_{cq}(X) = \sum_{i=0}^{d_A-1} \text{Tr}(P|\varphi_i\rangle\langle\varphi_i|_A) \sigma_{i,B} \quad (*)$$

For each $\sigma_{i,B}$, apply the spectral theorem:

$$\sigma_{i,B} = \sum_{j=0}^{d_B-1} \lambda_{j,i} |\psi_{j,i}\rangle\langle\psi_{j,i}|_B$$

Rewrite (*):

$$L_{cq}(X) = \sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} \text{Tr}(X|\varphi_i\rangle\langle\varphi_i|_A) \lambda_{j,i} |\psi_{j,i}\rangle\langle\psi_{j,i}|_B$$

Notice that:

$$\begin{aligned} \text{Tr}(X|\varphi_i\rangle\langle\varphi_i|_A) \lambda_{j,i} |\psi_{j,i}\rangle\langle\psi_{j,i}|_B &= \lambda_{j,i}^{1/2} \langle\psi_{j,i}, \cdot\rangle (\lambda_{j,i}^{1/2} \langle\varphi_i, X\varphi_i\rangle_A) |\psi_{j,i}\rangle \\ &= \lambda_{j,i}^{1/2} \langle\varphi_i, \lambda_{j,i}^{1/2} \langle\psi_{j,i}, \cdot\rangle_B X\varphi_i\rangle_A |\psi_{j,i}\rangle \\ &= \lambda_{j,i}^{1/2} \langle\varphi_i, X(\lambda_{j,i}^{1/2} \langle\psi_{j,i}, \cdot\rangle_B \varphi_i)\rangle_A |\psi_{j,i}\rangle \\ &= V_{i,j,i} X V_{i,j,i}^\dagger \end{aligned}$$

where $V_{i,j,i} : \mathcal{H}_A \rightarrow \mathcal{H}_B$ and $V_{i,j,i}^\dagger : \mathcal{H}_B \rightarrow \mathcal{H}_A$ are given by:

$$\begin{aligned} V_{i,j,i} &= \lambda_{j,i}^{1/2} \langle\varphi_i, \cdot\rangle_A |\psi_{j,i}\rangle \\ V_{i,j,i}^\dagger &= \lambda_{j,i}^{1/2} \langle\psi_{j,i}, \cdot\rangle_B |\varphi_i\rangle \end{aligned}$$

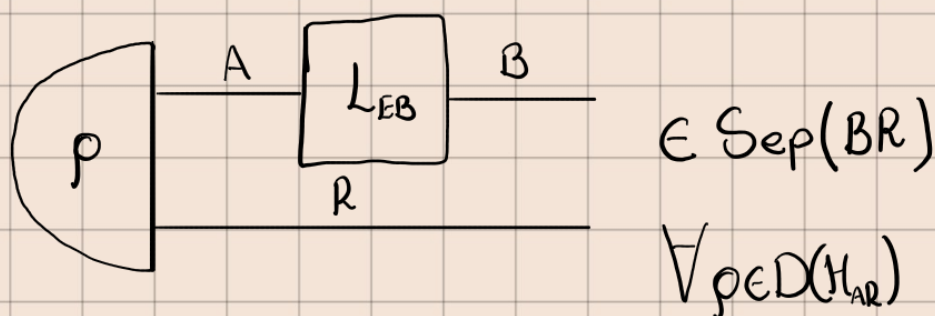
Straightforward to show that:

$$\bullet V_{j,i} X V_{j,i}^\dagger |u\rangle_B = T_{\Pi} (X |\varphi_i\rangle\langle\varphi_i|_A) \lambda_{j,i} |\psi_j\rangle\langle\psi_j|_B = \sum_{i,j} V_{j,i} X V_{j,i}^\dagger = L_{\text{eq}}(X)$$

$$\bullet \sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} V_{j,i}^\dagger V_{j,i} = \Pi_A$$

Hence $\{V_{j,i}\}_{i,j}$ is a collection of Kraus operators for L_{eq} .

4.6.2 A channel $L_{\text{EB}}: \text{Lin}(\mathcal{H}_A) \rightarrow \text{Lin}(\mathcal{H}_B)$ is **entanglement-breaking** if for any system $\mathcal{H}_R$:



Claim: if the Choi operation $J(L_{\text{EB}})$ is separable, then L_{EB} is entanglement-breaking.

$$\bullet J(L_{\text{EB}}) = (\text{id}_R \otimes L_{\text{EB}}) |\Gamma\rangle\langle\Gamma|_{RA} = d_A (\text{id}_R \otimes L_{\text{EB}}) |\Phi\rangle\langle\Phi|_{RA}$$

$$= \sum_{i,j=0}^{d_A-1} |i\rangle\langle j|_R \otimes L_{\text{EB}}(|i\rangle\langle j|_A)$$



$$\bullet L_{\text{EB}}(|i\rangle\langle j|_A) = (|i\rangle\langle i|_B \otimes \mathbb{I}_B) J(L_{\text{EB}}) (|j\rangle\langle j| \otimes \mathbb{I}_B)$$

$$= \sum_{k=1}^m p_k |\psi_k\rangle\langle\psi_k|_R \otimes |\varphi_k\rangle\langle\varphi_k|_B$$

$$\bullet (\text{id}_R \otimes L_{\text{EB}})(\rho_{RA}) = \sum_{i,j=0}^{d_A-1} |i\rangle\langle j| \otimes L_{\text{EB}}(X_{ij})$$

$$\sum_{i,j} |i\rangle\langle j| \otimes X_{ij} = \sum_{i,j=0}^{d_A-1} \sum_{n=1}^m (X_{ij})_{nn} |n\rangle\langle n|_A$$

$$\begin{aligned}
& \sum_{i,j} \sum_{\alpha,\beta} \sum_{k=1}^m p_k(x_{i,j}) \underbrace{\langle \Pi | \Psi_k \rangle \langle \Psi_k | \Pi \rangle}_{\text{Tr}(\chi_{i,j}^T |\Psi_k\rangle \langle \Psi_k|)} |\chi_{i,j}\rangle_R \langle \Psi_k | \Psi_k \rangle_B \\
& = \sum_{k=1}^m p_k \gamma_k \otimes |\Psi_k\rangle \langle \Psi_k|_B; \quad \gamma_k \equiv \sum_{i,j=0}^{d_A-1} \text{Tr}(\chi_{i,j}^T |\Psi_k\rangle \langle \Psi_k|) |\chi_{i,j}\rangle_R
\end{aligned}$$

$$\begin{aligned}
\bullet \langle u | \gamma_k | v \rangle_R &= \sum_{i,j} \text{Tr}(\chi_{i,j}^T |\Psi_k\rangle \langle \Psi_k|_R) \text{Tr}(|\chi_{i,j}\rangle \langle u | \chi | v \rangle_R) \\
&= \sum_{i,j} \text{Tr} \left[\chi_{i,j}^T |\Psi_k\rangle \langle \Psi_k| \otimes |\chi_{i,j}\rangle \langle u | \chi | v \rangle \right] \\
&= \sum_{i,j} \text{Tr} \left[(\underbrace{\chi_{i,j}^T}_{\geq 0} \otimes \underbrace{|\chi_{i,j}\rangle \langle u | \chi | v \rangle}_{\geq 0}) (\underbrace{|\Psi_k\rangle \langle \Psi_k|}_{\geq 0} \otimes \underbrace{|u\rangle \langle v|}_{\geq 0}) \right] \\
&= \text{Tr} \left[\underbrace{p_{RA}}_{\geq 0} \underbrace{|\Psi_k\rangle \langle \Psi_k|^* \otimes |u\rangle \langle v|}_{\geq 0} \right] \geq 0
\end{aligned}$$

Define $M_k \equiv p_k |\Psi_k\rangle \langle \Psi_k|$. Since $\mathcal{J}(L_{EB})$ is positive, L_{EB} is CPTP.

$$\bullet \text{Tr}(\chi_A) = \text{Tr}(L_{EB}(\chi_A)) \quad \forall \chi \quad \Rightarrow \text{Tr}(\chi_A) = \sum_{k=1}^m \text{Tr}(\chi_A M_k) \overbrace{\text{Tr}(|\Psi_k\rangle \langle \Psi_k|)}^1$$

$$\Rightarrow \text{Tr} \left[\left(\mathbb{I}_A - \sum_{k=1}^m M_k \right) \chi_A \right] \quad \forall \chi_A$$

$$L_{EB}(\chi) = \sum_k \text{Tr}(M_k \chi) |\Psi_k\rangle \langle \Psi_k|_B \quad \Rightarrow \boxed{\sum_{k=1}^m M_k = \mathbb{I}_A}$$

Consequently, L_{EB} is a measure-and-prepare channel. From this, we have (continuation of the calculations)

$$\text{id}_R \otimes L_{EB}(\rho_{RA}) = \sum_{k=1}^m q_k \tilde{\gamma}_k \otimes |\Psi_k\rangle \langle \Psi_k|, \quad (\text{SEPARABLE})$$

$$\text{Where: } q_k = \text{Tr} \gamma_k \quad \text{and} \quad \gamma_k \equiv \sum_{i,j=0}^{d_A-1} \text{Tr}(\chi_{i,j} M_k) |\chi_{i,j}\rangle.$$



4.6.3

• Classical quantum channel: $L_{cq}(X) = \sum_{i=1}^{d_A} \langle \psi_i | X | \psi_i \rangle_A \sigma_{i,B}$, $\{|\psi_i\rangle\}$ basis

I already proved that it is EB in the previous exercise. I'll just rewrite a cleaner proof.

Write

$$\rho_{RA} = \sum_{i,j=1}^{d_A} |i\rangle\langle j|_R \otimes X_{i,j} \quad (R=A). \quad (1)$$

Then

$$\begin{aligned} L_{cq}(\rho_{RA}) &= \sum_{i,j=1}^{d_A} |i\rangle\langle j| \otimes \left(\sum_{k=1}^{d_A} \langle \psi_k | X_{i,j} | \psi_k \rangle \sigma_{k,B} \right) \\ &= \sum_{k=1}^{d_A} \left(\sum_{i,j=1}^{d_A} \langle \psi_k | X_{i,j} | \psi_k \rangle |i\rangle\langle j| \right) \otimes \sigma_{k,B} \\ &= \sum_{k=1}^{d_A} p_k \omega_{k,R} \otimes \sigma_{k,B} \quad (*) \end{aligned}$$

where

$$\begin{cases} \omega_{k,R} = \Omega_{k,R} / p_k \\ \Omega_{k,R} = \sum_{i,j=1}^{d_A} \langle \psi_k | X_{i,j} | \psi_k \rangle |i\rangle\langle j|_R \\ p_k = \text{Tr}(\Omega_{k,R}) \end{cases}$$

• Positivity of $\Omega_{k,R}$:

$$\forall i,j: \langle v | (\langle \psi_k | X_{i,j} | \psi_k \rangle) |i\rangle\langle j| | v \rangle_R = \text{Tr}(X_{i,j} | \psi_k \rangle \langle \psi_k |_A) \text{Tr}(|i\rangle\langle j|_R | v \rangle \langle v|_R)$$

$$= \text{Tr}[(|i\rangle\langle i|_R \otimes X_{ij})(|v\rangle\langle v|_R \otimes |\psi_k\rangle\langle\psi_k|_A)]$$

Consequently:

$$\begin{aligned} \langle v | \Omega_{k,R} | v \rangle_R &= \text{Tr} \left[\sum_{i,j=1}^{d_A} (|i\rangle\langle i|_R \otimes X_{ij}) (|v\rangle\langle v|_R \otimes |\psi_k\rangle\langle\psi_k|_A) \right] \\ &= \text{Tr} [P_{RA} (|v\rangle\langle v|_R \otimes |\psi_k\rangle\langle\psi_k|_A)] \geq 0. \end{aligned}$$

$$\text{Thus, } \Omega_{k,R} \geq 0.$$

• P_k is bounded:

Since $\Omega_{k,R}$ is positive, its trace is non-negative, i.e.

$$P_k \geq 0.$$

Now:

$$\begin{aligned} \sum_{k=1}^{d_A} P_k &= \sum_{k=1}^{d_A} \text{Tr}(\Omega_{k,R}) \\ &= \sum_{k=1}^{d_A} \left(\sum_{\pi=1}^{d_A} \langle \pi | \Omega_{k,R} | \pi \rangle_R \right) \\ &= \sum_{k=1}^{d_A} \sum_{\pi=1}^{d_A} \text{Tr} [P_{RA} (|\pi\rangle\langle\pi|_R \otimes |\psi_k\rangle\langle\psi_k|_A)] \\ &= \text{Tr} \left[P_{RA} \left(\sum_{\pi=1}^{d_A} |\pi\rangle\langle\pi|_R \otimes \sum_{k=1}^{d_A} |\psi_k\rangle\langle\psi_k|_A \right) \right] \\ &= \text{Tr} [P_{RA} (\Pi_R \otimes \Pi_A)] = 1. \end{aligned}$$

Thus, $P_k \in [0,1]$ for each k .

Due to these, it follows that $w_{k,R} \in D(\mathcal{H}_R)$ and $\text{id}_R \otimes L_{\psi_k}(P_{RA})$ is a separable state. Conclusion: L_{ψ_k} is entanglement-breaking. $\blacksquare$

• Quantum-classical channel: $L_{qc}(X) = \sum_{i=1}^{d_B} \text{Tr}(X E_i) |\varphi_i\rangle\langle\varphi_i|_B$; $\{|\varphi_i\rangle\}_i$ ONB, $\{E_i\}_i$ POVM

Using eq. (1):

$$\begin{aligned} \text{id}_R \otimes L_{qc}(P_{RA}) &= \sum_{i,j=1}^{d_A} |\varphi_i\rangle\langle\varphi_j|_R \otimes \left(\sum_{k=1}^{d_B} \text{Tr}(X_{ij} E_k) |\varphi_k\rangle\langle\varphi_k|_B \right) \\ &= \sum_{k=1}^{d_B} \left(\sum_{i,j=1}^{d_A} \text{Tr}(X_{ij} E_k) |\varphi_i\rangle\langle\varphi_j|_R \right) \otimes |\varphi_k\rangle\langle\varphi_k|_B \\ &= \sum_{k=1}^{d_B} p_k \omega_{k,R} \otimes |\varphi_k\rangle\langle\varphi_k|_B \end{aligned}$$

where $\omega_{k,R}$ and p_k are defined analogously to the L_{qc} case.

Again, $\omega_{k,R} \in \text{ED}(\mathcal{H}_R)$ and $\{p_k\}_k$ is a prob. dist. Hence, $\text{id}_R \otimes L_{qc}(P_{RA})$ is separable and L_{qc} is EB. $\square$

4.7.1 and 4.7.2

Exercise 4.7.1 is just a particular case of 4.7.2, so we will solve the latter first.

• Pauli qubit Channel: $L^P(\rho) = \sum_{i,j=0}^1 p(i,j) Z^i X^j \rho X^j Z^i$, where

$p(i,j)$ is some prob. dist.

Since $ZX = iY$ and $Z^0 X^0 = X^0 Z^0 = \Pi_2$:

$$L^P(\rho) = p_{\Pi} \rho + p_X X \rho X + p_Y Y \rho Y + p_Z Z \rho Z. \quad (*)$$

Any qubit state ρ can be written as

$$\rho = \frac{1}{2} (\Pi_2 + \pi_X X + \pi_Y Y + \pi_Z Z). \quad (**)$$

Plugging $(**)$ into $(*)$ yields

$$\begin{aligned} L^P(\rho) &= \frac{p_0}{2} (\mathbb{I}_2 + \pi_x X + \pi_y Y + \pi_z Z) \\ &\quad + \frac{p_x}{2} X (\mathbb{I}_2 + \pi_x X + \pi_y Y + \pi_z Z) X \\ &\quad + \frac{p_y}{2} Y (\mathbb{I}_2 + \pi_x X + \pi_y Y + \pi_z Z) Y \\ &\quad + \frac{p_z}{2} Z (\mathbb{I}_2 + \pi_x X + \pi_y Y + \pi_z Z) Z \end{aligned}$$

$$= \frac{1}{2} \left[\mathbb{I}_2 + (p_0 + p_x - p_y - p_z) \pi_x X + (p_0 - p_x + p_y - p_z) \pi_y Y + (p_0 - p_x - p_y + p_z) \pi_z Z \right]$$

$\left\{ \begin{array}{lll} YXY = -X & XYX = -Y & XZX = -Z \\ ZXZ = -X & ZYZ = -Y & YZY = -Z \end{array} \right.$

The dephasing channel is obtained by setting $p_x = p_y = 0$, $p_z = p$ and $p_0 = 1 - p$.

4.7.3 Set $p_0 = p_x = p_y = p_z = 1/4$ in L^P . Then, all terms inside parentheses cancel each other and we obtain

$$\frac{1}{4} \rho + \frac{1}{4} X \rho X + \frac{1}{4} Y \rho Y + \frac{1}{4} Z \rho Z = \mathbb{I}/2.$$

4.7.4 Depolarizing channel: $L^P(\rho) = (1-p)\rho + p\mathbb{I}/d$.

In the qubit case: $L^P(\rho) = (1-p)\rho + p\left(\frac{1}{4}\rho + \frac{1}{4}X\rho X + \frac{1}{4}Y\rho Y + \frac{1}{4}Z\rho Z\right)$

$$= \left(1 - \frac{3p}{4}\right)\rho + p\left(\frac{1}{4}X\rho X + \frac{1}{4}Y\rho Y + \frac{1}{4}Z\rho Z\right).$$

4.7.5

The action of L^p on the Bloch vector:

$$L^p(\rho) = (1-p) \frac{1}{2} (\mathbb{I} + n_x X + n_y Y + n_z Z) + p \frac{\mathbb{I}}{2} = \frac{1}{2} (\mathbb{I} + \underbrace{(1-p) \vec{n} \cdot \vec{\sigma}}_{\text{Shrunk Bloch Vector}}).$$

4.7.6

Qudit twin. Recall that

$$\begin{cases} Z(y)|j\rangle = e^{i2\pi y j/d} |j\rangle & (\text{Phase operation}) \\ X(x)|j\rangle = |x \oplus j\rangle & (\text{Shift operation}) \end{cases}$$

We must show that

$$\frac{1}{d^2} \sum_{i,j=0}^{d-1} Z(i) X(j) \rho X^\dagger(j) Z^\dagger(i) = \mathbb{I}/d.$$

$$\bullet |\tilde{m}\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} e^{i2\pi mn/d} |n\rangle$$

$$\bullet |\tilde{m}\rangle \langle \tilde{m}|_K \langle \tilde{l}|\tilde{m}\rangle \langle \tilde{m}| = \frac{1}{d} e^{-i2\pi mK/d} e^{i2\pi mK/d} |\tilde{m}\rangle \langle \tilde{m}| \quad \bullet X(j)|\tilde{m}\rangle = e^{-i2\pi mj/d} |\tilde{m}\rangle$$

$$\bullet X(j) \rho X^\dagger(j) = \sum_{\tilde{m}, \tilde{n}=0}^{d-1} \rho_{\tilde{m}\tilde{n}} e^{-i2\pi m j/d} e^{i2\pi n j/d} |\tilde{m}\rangle \langle \tilde{n}|$$

$$\bullet Z(i) |\tilde{m}\rangle \langle \tilde{n}| Z^\dagger(i) = \frac{1}{d} \sum_{k,l=0}^{d-1} e^{i2\pi m k/d} e^{-i2\pi n l/d} e^{i2\pi i k/d} e^{-i2\pi i l/d} |k\rangle \langle l|$$

$$\bullet Z(i) X(j) \rho X^\dagger(j) Z^\dagger(i) = \frac{1}{d} \sum_{\tilde{m}, \tilde{n}=0}^{d-1} \rho_{\tilde{m}\tilde{n}} e^{-i2\pi m j/d} e^{i2\pi n j/d} \left(\sum_{k,l=0}^{d-1} e^{i2\pi m k/d} e^{-i2\pi n l/d} e^{i2\pi i k/d} e^{-i2\pi i l/d} |k\rangle \langle l| \right)$$

Summing over i only involves $e^{i2\pi i k/d} e^{-i2\pi i l/d} = e^{i2\pi i(k-l)/d}$, which will result in $d\delta_{kl}$.
Similarly, summing over j results in $d\delta_{mn}$. Thus:

$$\bullet \sum_{i,j=0}^{d-1} Z(i) X(j) \rho X^\dagger(j) Z^\dagger(i) = \sum_{\tilde{m}=0}^{d-1} \rho_{\tilde{m}\tilde{m}} \left(\sum_{k=0}^{d-1} |k\rangle \langle k| \right) d^2 = d^2 \cdot \mathbb{I}/d$$

$$\text{Hence: } \frac{1}{d^2} \sum_{i,j=0}^{d-1} Z(i) X(j) \rho X^\dagger(j) Z^\dagger(i) = \mathbb{I}/d. \quad \blacksquare$$